(1a) Only assuming the axioms and basic definitions, prove that \( a \mid b \) and \( a \mid c \) implies \( a \mid b + c \) for all integers \( a, b, c \in \mathbb{Z} \). Justify every step.

(1b) Write out all axioms you used in the proof of (1a) as mathematical statements.

(2) Prove that “\( P \) and \( Q \Rightarrow R \)” is equivalent to “\( P \Rightarrow R \) or \( Q \Rightarrow R \).”

(3) Prove that for all positive integers \( n \), \( 3 \mid 4^n + 5 \).

(4) Let \( A, B, C \) be sets. Prove that \( (A \cap C) \setminus B = (A \setminus B) \cup C \).

(5) Let \( A, B, C \) be sets. Prove that \( (A \setminus B) \cap C = \emptyset \) iff \( A \cap C \subseteq B \).

(6) Determine whether the following is true or false. If it is true, prove it. If it is false, disprove it.

Let \( X, Y \) be sets. \( \mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y) \).

(7) Let \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) be the following functions:

\[
\begin{align*}
    f(x) &= \begin{cases} 
    x + 2 & \text{if } x < -1 \\
    -x & \text{if } -1 \leq x \leq 1 \\
    x - 2 & \text{if } x > 1 
    \end{cases} \\
    g(x) &= \begin{cases} 
    x - 2 & \text{if } x < -1 \\
    -x & \text{if } -1 \leq x \leq 1 \\
    x + 2 & \text{if } x > 1 
    \end{cases}
\end{align*}
\]

Describe \( f \circ g \) and \( g \circ f \). Is \( f \) injective and/or surjective? Is \( g \) injective and/or surjective? Justify your answers with proofs.

(8) Let \( X, Y \) be sets and let \( f : X \to Y \) be a function. Recall that \( I_Y : Y \to Y \) denotes the identity function on \( Y \), sending \( y \mapsto y \). Prove that there exists a function \( g : Y \to X \) such that \( f \circ g = I_Y \) iff \( f \) is surjective.