1. Prove that for all integers \( a, b, \) and \( c, \)

\[(a \text{ divides } b) \text{ and } (b \text{ divides } c) \implies (a \text{ divides } c).\]

**Solution:** Let \( a, b, c \in \mathbb{Z} \) and suppose that \( a \) divides \( b \) and \( b \) divides \( c \). Because \( a \) divides \( b \), there is an integer \( n \in \mathbb{Z} \) such that \( b = na \), and because \( b \) divides \( c \) there is an integer \( m \in \mathbb{Z} \) such that \( c = mb \). But then

\[c = mb = m(na) = (mn)a.\]

Because \( m \) and \( n \) are integers, \( mn \) is also an integer, so we see that \( a \) divides \( c \).

2. Prove that 0 divides an integer \( a \) if and only if \( a = 0 \).

**Solution:** We have two implications to show: we need to show that \((0 \text{ divides } a \implies a = 0)\), and that \((a = 0 \implies 0 \text{ divides } a)\).

First we show that if \( 0 \) divides \( a \), then \( a = 0 \): if \( 0 \) divides \( a \), then by definition there is an integer \( k \in \mathbb{Z} \) such that \( a = 0k \). We also know from the last homework that \( 0k = 0 \), so

\[a = 0k = 0.\]

Now we show that if \( a = 0 \) then \( 0 \) divides \( a \): but if \( a = 0 \) then \( a = 0 \cdot 1 \), which shows that \( 0 \) divides \( a \).

3. Prove, from the inequality axioms, that if \( a, b, c \) are real numbers with \( a > 0 \), then

\[b \geq c \implies ab \geq ac.\]

**Solution:** We can prove this by proving the contrapositive: the contrapositive of this statement is

\[\text{not } ab \geq ac \implies \text{not } b \geq c.\]

Now recall that for real numbers \( x \) and \( y, x \geq y \) means \( x > y \) or \( x = y \). By the trichotomy law, we must have either \( x < y, x = y, \) or \( x > y \) for any two real numbers \( x, y \). Thus, we have

\[\text{not } x \geq y \iff x < y \quad \text{for real numbers } x, y.\]

Thus the contrapositive statement we need to prove is

\[ab < ac \implies b < c,\]

and this statement is true by the multiplication law because \( a > 0 \).
4. Prove that for negative real numbers $a, b, a < b \Rightarrow a^2 > b^2$.

**Solution:** Suppose $a$ and $b$ are negative real numbers, i.e. $a < 0$ and $b < 0$. Suppose furthermore that $a < b$; we will show that $a^2 > b^2$. Using the multiplication law, because $a < b$ and $a < 0$ we have $a^2 > ab$. Again using the multiplication law for $a < b$ and $b < 0$, we have $ab > b^2$. Thus we have

$$a^2 > ab \quad \text{and} \quad ab > b^2,$$

which implies $a^2 > b^2$ by the transitive law.

5. Proposition 3.2.1 states that $a < b$ is a sufficient condition for $4ab < (a + b)^2$. Is the condition also necessary? If so, prove it. If not, find a necessary and sufficient condition.

**Solution:** The condition is not necessary, for instance, if we take $a = 1$ and $b = 0$, then $4ab < (a + b)^2$ but it is not the case that $a < b$. A necessary and sufficient condition is given by $a \neq b$: in fact, the sequence of steps in the proof of Proposition 3.2.1 shows that if $a \neq b$ then $4ab < (a + b)^2$. Furthermore, to prove the other direction, one can note that each one of these steps is reversible: that is, we have implications

$$4ab < (a + b)^2 \implies a^2 + 2ab + b^2 \implies 0 < a^2 - 2ab + b^2 \implies 0 < (a - b)^2 \implies a - b \neq 0 \implies a \neq b.$$
6. Prove by contradiction that there do not exist integers \( m \) and \( n \) such that \( 14m + 21n = 100 \).

**Solution:** Suppose for a contradiction that there exist \( m, n \in \mathbb{Z} \) such that \( 14m + 21n = 100 \). Then

\[
100 = 7(2m + 3n),
\]

which implies that 7 divides 100 because \( 2m + 3n \) is an integer, but this is a contradiction because 7 does not divide 100.

7. Prove by contradiction that for any integer \( n \),

\[ n^2 \text{ is odd} \implies n \text{ is odd}. \]

**Solution:** Suppose for a contradiction that there is an integer \( n \) such that \( n^2 \) is odd, but \( n \) is even. Writing \( n = 2k \) for an integer \( k \), we have

\[
n^2 = (2k)^2 = 4k^2 = 2(2k^2).
\]

Because \( 2k^2 \) is an integer this shows \( n^2 \) is even, which is a contradiction because we assumed that \( n^2 \) is odd, and no integer can be both odd and even.

8. Prove the result of the previous exercise by writing down its contrapositive.

**Solution:** The contrapositive of the statement

\[ n^2 \text{ is odd} \implies n \text{ is odd} \]

is the statement

\[ n \text{ is even} \implies n^2 \text{ is even}. \]

This can be proved in a similar vein to the previous problem: if \( n \) is even then we can write \( n = 2k \) for some \( k \in \mathbb{Z} \), and then notice that \( n^2 = 4k^2 = 2(2k^2) \), which is even because \( 2k^2 \in \mathbb{Z} \).
9. Prove that, for all real numbers \(a\) and \(b\),

\[ |a + b| \leq |a| + |b|. \]

Give a necessary and sufficient condition for equality.

**Solution:** We use the fact that for real numbers \(x, y, |x| \leq |y| \iff x^2 \leq y^2 \) (this is problem 4.6). Because \(|a| + |b| \geq 0\), we have that \(||a| + |b|| = |a| + |b|\), and because \(|a + b|^2 = (a + b)^2\),

\[ |a + b| \leq |a| + |b| \iff (a + b)^2 \leq (|a| + |b|)^2. \]

Expanding both sides and using the fact that \(2ab \leq 2|a||b|\), we get

\[ (a + b)^2 = a^2 + 2ab + b^2 = |a|^2 + 2ab + |b|^2 \leq |a|^2 + 2|a||b| + |b|^2 = (|a| + |b|)^2. \]

Now, we need to find a necessary and sufficient condition for equality. Because \(|x| = |y| \iff x^2 = y^2\) for real numbers \(x, y\), we have that

\[ |a + b| = |a| + |b| \iff (a + b) = (|a| + |b|)^2. \]

Expanding both terms in the right hand side again, this is equivalent to the condition

\[ a^2 + 2ab + b^2 = |a|^2 + 2|a||b| + |b|^2. \]

By using the fact that \(a^2 = |a|^2\) and \(b^2 = |b|^2\), subtracting \(a^2 + b^2\) from both sides tells us this is equivalent to the condition \(2ab = 2|a||b|\) which, dividing by 2, this is equivalent to the condition \(ab = |a||b|\). These values will be equal if either \(a\) or \(b\) is zero, or if the two have the same "sign", i.e. \(a < 0\) and \(b < 0\) or \(a > 0\) or \(b > 0\). Therefore

\[ |a + b| = |a|^2 + |b|^2 \iff (a \leq 0 \text{ and } b \leq 0) \text{ or } (a \geq 0 \text{ and } b \geq 0). \]

10. Define an identity element for multiplication. Prove that if there exists an identity element, then that element is unique.

**Solution:** We will say that \(e\) is an identity element for multiplication if \(ea = a = ae\) for all other values \(a\).

Suppose there exists an identity element \(e\). We need to show it is unique: that is, we need to show that if \(e'\) is another identity element, then \(e = e'\). But if \(e'\) is another identity element then using our definition of identity element, we have

\[ e = ee' \quad \text{and} \quad ee' = e', \]

which then implies \(e = e'\).