1. Let $X = \{a, b, c\}$ and $Y = \{d, e\}$. What is the cardinality of the Cartesian product $X \times Y$? Write down an explicit bijection $\mathbb{N}_n \to X \times Y$ where $n = |X \times Y|$.

**Solution:** Because $|X| = 3$ and $|Y| = 2$, the using the multiplication principle (Theorem 10.2.3) we get

$$|X \times Y| = |X||Y| = 6.$$ 

Explicitly, we have a bijection $f : \mathbb{N}_6 \to X \times Y$ as indicated in the following table:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$f(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(a, d)</td>
</tr>
<tr>
<td>2</td>
<td>(a, e)</td>
</tr>
<tr>
<td>3</td>
<td>(b, d)</td>
</tr>
<tr>
<td>4</td>
<td>(b, e)</td>
</tr>
<tr>
<td>5</td>
<td>(c, d)</td>
</tr>
<tr>
<td>6</td>
<td>(c, e)</td>
</tr>
</tbody>
</table>

2. Each of a collection of 144 tiles is either triangular or square, either red or blue, and either wooden or plastic. Given that there are 68 wooden tiles, 69 red tiles, 75 triangular tiles, 36 red wooden tiles, 40 triangular wooden tiles, 38 red triangular tiles, and 23 red wooden triangular tiles, how many blue plastic square tiles are there?

**Solution:** We let $X$ be the set of tiles, and let $A, B, C$ be the subsets consisting of wooden tiles, red tiles, and triangular tiles respectively. The number of tiles in each of these sets and their intersections is summarized by the following table:

| $Y$ | $|Y|$ |
|-----|------|
| $A$ | 68   |
| $B$ | 69   |
| $C$ | 75   |
| $A \cap B$ | 36   |
| $A \cap C$ | 40   |
| $B \cap C$ | 38   |
| $A \cap B \cap C$ | 23   |

Now using the inclusion-exclusion principle (Proposition 10.3.2), we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = 121.$$ 

Because $A \cup B \cup C$ denotes the set of tiles which are *either* wooden or red or triangular, the set of blue plastic square tiles is given by the complement of $A \cup B \cup C$ in $X$. Thus their number is equal to

$$|X \setminus (A \cup B \cup C)| = |X| - |A \cup B \cup C| = 144 - 121 = 23,$$

and we see there are 23 such tiles.
3. Suppose that \( N_n \to X \) is a surjection. Prove by induction on \( n \) that \( X \) is finite and \( |X| \leq n \).

**Solution:** We prove the result by induction on \( n \):

**Base case:** Let \( n = 1 \). If \( |X| > 1 \) then choosing distinct \( x, x' \in X \), for any function \( f : N_1 \to X \) we cannot have \( f(1) \) equal to both \( x \) and \( x' \), and because 1 is the only element of \( N_1 \) we see that \( f \) cannot be injective. Taking the contrapositive, we see that if \( n = 1 \) then having a surjection \( N_1 \to X \) implies \( |X| \leq 1 \).

**Inductive step:** Suppose the result is true for \( n = k \); that is, we are assuming

For any set \( Y \), if there is a surjection \( N_k \to Y \) then \( |Y| \leq k \).

Now we show the result is true for \( n = k + 1 \): suppose we have a surjection \( f : N_{k+1} \to X \). We have two cases:

**Case 1:** Suppose every \( x \in X \) can be written as \( f(i) \) for some \( i \in N_k \). We define the restriction function \( f_{N_k} : N_k \to X \) given by \( f_{N_k}(i) := f(i) \in X \). By assumption, \( f_{N_k} \) is surjective, and therefore by the inductive hypothesis we get \( |X| \leq k \), which implies \( |X| \leq k + 1 \).

**Case 2:** Suppose there is \( x_0 \in X \) which cannot be written in the form \( f(i) \) for \( 1 \leq i \leq k \). This means that if \( i \in N_k \), then \( f(i) \neq x_0 \), so we can define a function \( g : N_k \to X \setminus \{x_0\} \) by \( g(i) := f(i) \in X \setminus \{x_0\} \). This is also a surjection: note because \( f \) is surjective, we can write \( x_0 = f(i) \) for some \( i \in N_{k+1} \), but because this is not possible for \( i \in N_k \) we conclude \( x_0 = f(k + 1) \). It follows that if \( x \in X \setminus \{x_0\} \), then choosing some \( i \in N_{k+1} \) with \( x = f(i) \), we do not have \( i = k + 1 \) since \( x \neq x_0 \), and thus \( i \in N_{k+1} \setminus \{k + 1\} = N_k \), and thus \( x = f(i) = g(i) \). This shows \( g \) is surjective, so we can apply the induction hypothesis to \( g : N_k \to X \setminus \{x_0\} \) to conclude (i) \( X \setminus \{x_0\} \) is finite, which implies \( X = (X \setminus \{x_0\}) \cup \{x_0\} \) is finite by Theorem 10.2.1, and (ii) we get \( |X \setminus \{x_0\}| \leq k \). But \( |X \setminus \{x_0\}| = |X| - 1 \), so we have

\[
|X| - 1 \leq k \implies |X| \leq k + 1.
\]

This completes the induction.

4. Suppose \( X \) and \( Y \) are finite sets such that \( X \subseteq Y \). Prove that

(i) \( X = Y \iff |X| = |Y| \),

(ii) \( X \subset Y \iff |X| < |Y| \).

**Solution:**

(i) Of course, if \( X = Y \) then \( |X| = |Y| \). On the other hand, if \( |X| = |Y| \) then we consider the inclusion map \( i : X \to Y \) (defined by \( i(x) = x \in Y \)). This map is injective, so because \( |X| = |Y| \), Theorem 11.1.7 implies that \( i \) is also a surjection. This implies
Y \subseteq X because if y \in Y there exists x \in X with i(x) = y, but i(x) = x so we have y = x \in X. Then because X \subseteq Y by assumption we conclude X = Y.

(ii) First we note that because X \subseteq Y we have |X| \leq |Y| (see Corollary 11.1.5). Thus

\[
X \subset Y \iff X \neq Y \quad \text{(because } X \subseteq Y) \\
\iff |X| \neq |Y| \quad \text{(by (i))} \\
\iff |X| < |Y| \quad \text{(because } |X| \leq |Y|).
\]

5. Prove by induction on n that if A is a set of positive integers without a least element then \(N_n \subseteq \mathbb{Z}^+ \setminus A\) for every n, so that A is the empty set. Deduce the well-ordering principle: every non-empty subset of positive integers has a least element.

**Solution:** Suppose A is a set of positive integers without a least element. We prove by induction on n that \(N_n \subseteq \mathbb{Z}^+ \setminus A\):

**Base case:** Note that if 1 \in A then 1 gives a least element of A, so because we are assuming A does not have a least element we must have 1 \not\in A. Because \(N_1 = \{1\}\) this precisely says that \(N_1 \subseteq \mathbb{Z}^+ \setminus A\), which proves the base case n = 1.

**Inductive step:** Suppose it is true for n = k, i.e. suppose \(N_k \subseteq \mathbb{Z}^+ \setminus A\). We show that \(N_{k+1} \subseteq \mathbb{Z}^+ \setminus A\); we need to show that i \not\in A for i = 1, \ldots, k + 1, but by the inductive hypothesis we already have i \not\in A for i = 1, \ldots, k, so we just need to show k + 1 \not\in A. For a contradiction, suppose k + 1 \in A; then we claim k + 1 is a least element of A. By definition of a least element, we need to show (i) k + 1 \in A, and (ii) if m \in A then k + 1 \leq m. We have assumed (i) is true by assumption, and furthermore, if m \in A then because \(N_k \subseteq \mathbb{Z}^+ \setminus A\), we know that m \not\in N_k. Because \(N_k = \{i \in \mathbb{Z}^+ \mid i \leq k\}\), this precisely means that m > k, i.e. m \geq k + 1. This shows k + 1 is a least element for A, giving a contradiction. We conclude k + 1 \not\in A and thus \(N_{k+1} \subseteq \mathbb{Z}^+ \setminus A\), completing the inductive step.

We now conclude by induction that \(N_n \subseteq \mathbb{Z}^+ \setminus A\) for all n \in \mathbb{Z}^+. But then, for any positive integer n, we have n \in N_n \subseteq \mathbb{Z}^+ \setminus A, and therefore n \not\in A. We conclude that no positive integer n belongs to A, so A is the empty set.

Finally, we can conclude the well-ordering principle. We just proved the following statement about a set A consisting of positive integers:

If A does not have a least element, then A is the empty set.

Taking the contrapositive, we have proved the following about a set A of positive integers:

If A is non-empty, then A has a least element.
6. Of the 182 students who are taking three first year core mathematics modules (Reasoning, Algebra and Calculus), 129 like Reasoning, 129 like Algebra, 129 like Calculus, 85 like Reasoning and Algebra, 89 like Reasoning and Calculus, 86 like Algebra and Calculus, and 54 like all three modules. How many students like none of the core modules?

Solution: Let $X$ be the set of students. Let $A$, $B$, and $C$ denote the subsets of students who like Reasoning, Algebra, and Calculus respectively. Then the following table lists the cardinality of each set and their respective intersections:

<table>
<thead>
<tr>
<th>$Y$</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$A \cap B$</th>
<th>$A \cap C$</th>
<th>$B \cap C$</th>
<th>$A \cap B \cap C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>Y</td>
<td>$</td>
<td>129</td>
<td>129</td>
<td>129</td>
<td>85</td>
<td>89</td>
</tr>
</tbody>
</table>

Now using Proposition 10.3.2 we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = 181.$$  

Because $A \cup B \cup C$ is the set of students who like at least one core module, we see the set of students who do not like any core module is given by $X \setminus (A \cup B \cup C)$. Because $|X| = 182$ and $|A \cup B \cup C| = 181$, we see there is exactly one such student.

7. Prove (by induction on $n$) the general inclusion-exclusion principle, states as follows:

Let $A_1, \ldots, A_n$ be finite sets. For $I \subseteq \mathbb{N}_n$, write

$$A_I := \bigcap_{i \in I} A_i.$$  

Then

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{\varnothing \neq I \subseteq \mathbb{N}_n} (-1)^{|I|-1}|A_I|.$$  

Solution: We prove the result by induction on $n$:

Base case: If $n = 1$ then this is merely the equality $|A_1| = |A_1|$. Note that the cases $n = 2$ and $n = 3$ are given by Proposition 10.3.1 and 10.3.2 respectively.

Inductive step: Suppose it is true for $n = k$; that is, suppose that for any collection of finite sets $A_1, \ldots, A_k$, defining $A_1$ as in the problem, we have the equality

$$\left| \bigcup_{i=1}^{k} A_i \right| = \sum_{\varnothing \neq I \subseteq \mathbb{N}_n} (-1)^{|I|-1}|A_I|.$$  

Then suppose we have finite sets $A_1, \ldots, A_{k+1}$. For $i \in \mathbb{N}_k$ we let $B_i := A_i \cap A_{k+1}$, and write $B_i$ in the same way for $1 \subseteq \mathbb{N}_k$, and note $B_i = A_i \cup (n)$ for all $\varnothing \neq I \subseteq \mathbb{N}_k$. Then we
can apply the induction hypothesis to $B_1, \ldots, B_k$ as well, and we get using Proposition 10.3.1 (i.e. the $n=2$ case)

\[
\left| \bigcup_{i=1}^{k+1} A_i \right| = \left| \left( \bigcup_{i=1}^{k} A_i \right) \cup A_{k+1} \right|
\]

\[
= \left| \bigcup_{i=1}^{k} A_i \right| + |A_{k+1}| - \left| \bigcup_{i=1}^{k} A_i \cap A_{k+1} \right|
\]

\[
= \left| \bigcup_{i=1}^{k} A_i \right| + |A_{k+1}| - \left| \bigcup_{i=1}^{k} B_i \right|
\]

\[
= \sum_{\emptyset \neq I \subseteq N_k} (-1)^{|I|-1}|A_I| + |A_{k+1}| - \sum_{\emptyset \neq I \subseteq N_k} (-1)^{|I|-1}|A_{I \cup \{k+1\}}|
\]

\[
= \sum_{\emptyset \neq I \subseteq N_{k+1}} (-1)^{|I|-1}|A_I|.
\]

The (hopefully) only mystery in these equality is the last one. But this is given by considering the non-empty $I \subseteq N_{k+1}$ and subdividing into three cases: the first sum in the above line corresponds to the $I$ which do not contain $k+1$, the middle term corresponds to the $I$ which only contain $k+1$, i.e. it corresponds exactly to $I = \{k+1\}$, and the last sum corresponds to the $I$ which contain $k+1$ and also more elements, so $I$ can be written as $I = I' \cup \{k+1\}$ where $\emptyset \neq I' \subseteq N_k$.

8. Suppose $A$ and $B$ are non-empty finite sets of real numbers such that $A \subseteq B$. Prove that

\[
\min B \leq \min A \leq \max A \leq \max B.
\]

**Solution:** Let $a, a'$ denote the minimum and maximum of $A$, respectively, and let $b, b'$ denote the minimum and maximum of $B$, respectively. We have three inequalities to show:

$b \leq a$: By definition of the minimum of $B$, if $x \in B$ then $b \leq x$; however, $a \in A$ by definition of the minimum of $A$, and because $A \subseteq B$ this implies $a \in B$, so $b \leq a$.

$a \leq a'$: By definition of maximum we must have $a' \in A$, but then by definition of the minimum we must have $a \leq a'$.

$a' \leq b'$: By definition of the maximum, if $x \in B$ then $x \leq b'$; but by definition of maximum we have $a' \in A$, which implies $a' \in B$ since $A \subseteq B$, so $a' \leq b'$.
9. What is wrong with the following argument purporting to prove that for every non-empty finite set \( A \) of real numbers, \( \min A = \max A \)?

**Proof.** We prove by induction on \( n \) that if \( |A| = n \) then \( \min A = \max A \).

**Base case:** If \( |A| = 1 \) then \( A \) is a singleton set \( A = \{a_1\} \) and \( \min A = a_1 = \max A \).

**Inductive step:** Suppose that for some positive integer \( k \) the result is true for \( n = k \). Let \( A \) be a set of real numbers with \( |A| = k + 1 \). Let \( a_1 = \min A \) and \( a_2 = \max A \). Then

\[
\begin{align*}
a_1 &= \min(A \setminus \{a_2\}) \\
&= \max(A \setminus \{a_2\}) \quad \text{(by inductive hypothesis)} \\
&\geq \max(A \setminus \{a_1, a_2\}) \quad \text{(by the previous problem)} \\
&\geq \min(A \setminus \{a_1, a_2\}) \quad \text{(by the previous problem)} \\
&\geq \min(A \setminus \{a_1\}) \quad \text{(by the previous problem)} \\
&= \max(A \setminus \{a_1\}) \quad \text{(by inductive hypothesis)} \\
&= a_2.
\end{align*}
\]

Hence \( a_1 \geq a_2 \) and since clearly \( a_1 \leq a_2 \) this proves that \( a_1 = a_2 \) as required to deduce the result for \( n = k + 1 \) and so proving the inductive step.

**Conclusion:** Hence by induction on \( n \), \( \min A = \max A \) for a set of cardinality \( n \) for all positive integers \( n \), and so for all finite sets.

**Solution:** The issue in the proof is that the inductive step does not successfully prove the implication \( P(1) \implies P(2) \): this is because if \( |A| = 2 \), then \( A \setminus \{a_1, a_2\} \) will be the empty set, and we cannot apply the previous problem (because the hypothesis of the problem required \( A \) and \( B \) be non-empty; in fact, no real number can even satisfy the definition for the minimum/maximum of the empty set).

10. Suppose that there is an injection \( f : \mathbb{Z}^+ \to X \). Prove by contradiction that \( X \) is an infinite set. [Use Corollary 11.1.1 noting that, for any \( n \geq 1 \), \( f \) restricts to given an injection \( \mathbb{N}_{n+1} \to X \).]

**Solution:** Suppose for a contradiction that \( X \) is a finite set. By definition this means \( |X| = n \) for some integer \( n \geq 1 \). However, note that for any \( k \in \mathbb{Z}^+ \), because \( \mathbb{N}_{k+1} \subseteq \mathbb{Z}^+ \) we can define the "restriction" of the function \( f \):

\[
f|_{\mathbb{N}_k} : \mathbb{N}_k \to X \text{ defined by } f|_{\mathbb{N}_k}(i) := f(i) \in X.
\]

Furthermore, this restriction function will be injective because \( f \) is injective by assumption; but now, take \( k = n + 1 \in \mathbb{Z}^+ \). Because this restriction \( f|_{\mathbb{N}_{n+1}} : \mathbb{N}_{n+1} \to X \) is injective, Corollary 11.1.1 implies that \( n + 1 \leq |X| = n \), which is a contradiction. We conclude \( X \) is an infinite set.