What is the degree of a polynomial?

- Algebraic Operations on Polynomials
- Zeros of a Polynomial
- Asymptotic Behavior of Polynomials

**Def:** A polynomial is a function $p$ such that

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $n \geq 0$ is an integer and $a_0, a_1, \ldots, a_n$ are constant real numbers.

**Ex:** $p(x) = 3x^2 + x - 10$

**Ex:** $p(x) = -\frac{5}{2} x^9 + \sqrt{2} x^5 - 3x$
Def: If \( a_n \neq 0 \), then the degree of the polynomial, 
\[ p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 x + a_0 \]
is \( n \).

\[ \text{Ex: } p(x) = x^2 - 3 \quad (\text{degree } 2) \]

\[ \text{Ex: } p(x) = 2 \quad (\text{degree } 0) \]

\[ \text{Ex: } p(x) = -10 x^8 + x^3 + 1 \quad (\text{degree } 8) \]

Let \( P, Q \) be polynomials, then
\[ (P + Q)(x) := p(x) + q(x) \]
\[ (P - Q)(x) := p(x) - q(x) \]
\[(p \cdot q)(x) := p(x)q(x)\]

Note: \(p+q, p-q,\) and \(pq\) are new polynomials. (Polynomials are closed under addition, subtraction, and multiplication.)

\[\exists x: \text{ Let } p(x) = x^2 + 3x - 1, \quad q(x) = x^4 + 5x^3 - 3x, \text{ then find } (p+q)(x), (p-q), \text{ and } pq, \text{ and state the degree.} \]

\[(p+q)(x) = p(x) + q(x) = (x^2 + 3x - 1) + (x^4 + 5x^3 - 3x) = x^4 + 5x^3 + x^2 - 1 \text{ (degree 4)}\]
\((p-q)(x) = p(x) - q(x) = (x^2 + 3x - 1) - (x^4 + 5x^3 - 3x) = -x^4 - 5x^3 + x^2 + 6x - 1\) (degree 4)

\((pq)(x) = p(x)q(x) = (x^2 + 3x - 1)(x^4 + 5x^3 - 3x) = (x^2)(x^4 + 5x^3 - 3x) + 3x(x^4 + 5x^3 - 3x) - 1(x^4 + 5x^3 - 3x) = x^6 + 5x^5 - 3x^3 + 3x^5 + 15x^4 - 9x^2 - x^4 - 5x^3 + x = x^6 + 8x^5 + 14x^4 - 8x^3 - 9x^2 + 3x\) (degree 6)

**Definition:** A number \(r\) such that \(p(r) = 0\) is called a zero of \(p\).

Let \(p(x) = ax^2 + bx + c\), then the zeros of \(p\) are given by

\[
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]
Ex: (Factoring) \( p(x) = x^2 - 1 \)

\( \Rightarrow p(x) = (x + 1)(x - 1) \)

\((x+1)\) and \((x-1)\) are called factors of \( p \), and note that \( x + 1 = 0 \Rightarrow x = -1 \) is a zero of \( p \) and \( x - 1 = 0 \Rightarrow x = 1 \) is a zero of \( p \).

Ex: \( p(x) = (x - 3)(x + 2) \)

then \( x - 3 \), \( x + 2 \) and \( x \) are factors of \( p \), and \( 3, -2, \) and \( 0 \) are zeros of \( p \).
Ex: Find a polynomial with zeros $1, -3, 2, \frac{1}{2}$.

$p(x) = (x-1)(x+3)(x-2)(x-\frac{1}{2})$

(What is the degree of $p$?)

- A polynomial of degree $n$ has at most $n$ zeros.

Ex: $p(x) = x^2 + 1$ is degree 2 and has NO zeros.

Ex: $p(x) = x^3 - x = x(x^2 - 1) = x(x+1)(x-1)$

$p$ is degree 3 and has 3 zeros.
To determine whether a polynomial is positive or negative as $x$ approaches $\pm \infty$, factor out the term with the highest degree.

\[ p(x) = -2x^5 + 10x^4 + 100 \]

\[ = -2x^5 (1 - 5x^{-1} - 50x^{-5}) \]

$\Rightarrow p$ is negative as $x$ approaches infinity.
Ex: Is \( p(x) = -2x^5 + 10x^4 + 100 \) positive or negative as \( x \) approaches negative infinity?

\[
p(x) = -2x^5 + 10x^4 + 100 = -2x^5 \left(1 - \frac{5}{x} - \frac{50}{x^5}\right)
\]

Notice that if \( x \) is negative, then \( x^5 \) is negative and \(-2x^5\) is positive. Therefore, \( p \) is positive as \( x \) approaches negative infinity.

If \( a, b \) are real numbers and \( p(a) < 0 \), \( p(b) > 0 \), then \( p \) has at least one zero in the interval \((a, b)\).
Ex: \( p(x) = x^4 - 10 \).

Then \[ \begin{align*}
  p(1) &= 1^4 - 10 = -9 < 0 \\
  p(2) &= 2^4 - 10 = 16 - 10 = 6 > 0
\end{align*} \]
and \( p \) has at least one zero on the interval \((1, 2)\).