• Fractions as exponents
• Exponential functions

\( y, x \in \mathbb{R} \), \( m, n \) are integers then,

\[
\begin{align*}
X^n X^m &= X^{m+n} \\
(x^n)^m &= X^{nm} \\
X^n y^n &= (xy)^n \quad \frac{x^m}{x^n} = x^{m-n} \\
x^0 &= 1 \quad (0^0 \text{ is undefined})
\end{align*}
\]

Let \( m \) be a rational number (i.e. a fraction of integers) and \( n \) is a positive integer.

What is \( x^m = ? \)

We would like \( (x^m)^n = x^{mn} \).

What if \( m = \frac{1}{n} \)?
\[ (x^m)^n = \left( x^{\frac{1}{n}} \right)^n = x^{\frac{n}{n}} = x^1 = x \]

\[ \sqrt[3]{8} = 2 \quad \text{since} \quad 2^3 = 8 \]

- We will call \( x^{\frac{1}{n}} \) the \( n \)th root of \( x \). Recall \( x^{\frac{1}{2}} = \sqrt{x} \) is the square root of \( x \).

\[ \sqrt[2]{-4} \]

\text{Ex: What is } (-4)^{\frac{1}{2}}? \]

If we want \((-4)^{\frac{1}{2}})^2 = -4\),
then \((-4)^{\frac{1}{2}} \text{ is the number that when squared equals -4. There are NO such real numbers that satisfy this!} \]
Def: \( m > 0 \) is an integer and \( x \in \mathbb{R} \), then \( x^{\frac{m}{n}} \) is the real number satisfying

\[
\left( x^{\frac{m}{n}} \right)^n = x
\]

such that

\[\begin{align*}
&\text{if } x < 0 \text{ and } m \text{ is even,} \\
&\text{then } x^{\frac{m}{n}} \text{ is undefined.}
\end{align*}\]

\[\text{Notation}\]

\[\begin{align*}
x^{\frac{1}{3}} &= \sqrt[3]{x} \quad \text{(cube root)} \\
x^{\frac{m}{n}} &= \sqrt[n]{x} \quad \text{(m\textsuperscript{th} root)}
\end{align*}\]

\[\text{Recall} \quad \sqrt{x} \sqrt{y} = \sqrt{xy} \quad \text{but} \quad \sqrt{x} + \sqrt{y} \neq \sqrt{x+y}.\]
We can think of $x^{\frac{1}{m}}$ as the inverse of $x^m$.

i.e., $f(x) = x^m$

$f^{-1}(y) = y^{\frac{1}{m}}$

Define $x^{m/n}$ where $m > 0$ and $n$ are integers by,

$$x^{m/n} := (x^{1/m})^n$$

Algebraic Properties of Exponents

Let $a, b$ be positive and $x, y \in \mathbb{R}$,

$$b^x b^y = b^{x+y}$$

$$(b^x)^y = b^{xy}$$

$$a^x b^x = (ab)^x$$

$$b^0 = 1$$

$$b^{-x} = \frac{1}{b^x}$$

$$\frac{a^x}{a^y} = a^{x-y}$$

$$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$$
Ex: \((25)^{\frac{3}{2}} = (25^{\frac{1}{2}})^3 = (\sqrt{25})^3 = 5^3 = 125\)

Ex: \((32)^{\frac{3}{5}} = (32^{\frac{1}{5}})^3 = (2^5)^{\frac{3}{5}} = 2^3 = 8\)

Ex: \((32^{-\frac{4}{5}} = (32^{\frac{1}{5}})^{-4} = 2^{-4} = \frac{1}{2^4} = \frac{1}{16}\)

Ex: \((-8)^{\frac{2}{3}} = (-8^{\frac{1}{3}})^2 = (-2^3)^{\frac{2}{3}} = (-2^2) = -4 = -128\)

**Exponential Function**

Let \(b > 0\), then the exponential function with base \(b\) is

\[ f(x) = b^x \]
Ex: Plot \( f(x) = 2^x \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>( 2^{-4} = \frac{1}{16} )</td>
</tr>
<tr>
<td>-2</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
</tr>
</tbody>
</table>

The value of \( f \) increases quickly for positive \( x \)!

Ex: Find \( b \) such that

\[ b^4 = 81, \]

\[ (b^4)^{\frac{1}{4}} = (81)^{\frac{1}{4}} \]

\[ b = 81^{\frac{1}{4}} = (3^4)^{\frac{1}{4}} = 3. \]