continuous compounding

Recall the formula for the amount owed after $t$ years, when $P_0$ was originally borrowed at an interest rate, $r$, compounded $n$ times per year,

$$P(t) = P_0 \left(1 + \frac{r}{n}\right)^{nt}.$$

As $n$ becomes very large,

$$P_0 \left(1 + \frac{r}{n}\right)^{nt} \approx P_0 (e^r)^t = P_0 \ e^{rt}.$$

This is known as continuous compounding.
Ex: If you borrow $10,000 compounded continuously at 6% annual interest, how much do you owe after 20 years?

\[ P(t) = P_0 e^{rt} \]

\[ = 10,000 e^{0.06t} \]

\[ P(20) = 10,000 e^{0.06 \cdot 20} \]

\[ = 10,000 e^{1.2} \]

\[ = 33,201.16 \]

Now assume you borrow \( P_0 \) compounded continuously at \( r \) interest annually. How long before you owe twice as much money?
\[ P(t) = P_0 e^{rt} \]
\[ 2P_0 = P_0 e^{rt} \]
\[ \Rightarrow 2 = e^{rt} \]
\[ \Rightarrow \ln(2) = rt \]
\[ \Rightarrow t = \frac{\ln(2)}{r} \]

**Ex:** You borrow $10,000 at 6\% interest compounded continuously. How long before you owe double the original amount borrowed.

\[ t = \frac{\ln(2)}{0.06} = 11.55 \text{ years} \]

**Ex:** How long before you owe five times as much?
\[ P(t) = 5P_0 = 5(10,000) = 50,000 \]
\[ 50,000 = 10,000 e^{0.06t} \]
\[ \Rightarrow 5 = e^{0.06t} \]
\[ \Rightarrow \ln(5) = 0.06t \]
\[ \Rightarrow t = \frac{\ln(5)}{0.06} \]
\[ = 26.82 \text{ years} \]

**Ex:** If you borrow $10,000 with continuous compounding, then for what interest rate will your debt double in 10 years?

\[ 20,000 = 10,000 e^{r(10)} \]
\[ \Rightarrow z = e^{r(10)} \]
\[ \Rightarrow \ln(z) = 10r \]
\[ \Rightarrow r = \frac{\ln(z)}{10} = 0.069 = 6.9\% \]
Solving Equations

- A solution to an equation is an assignment of values to the variables that satisfies the equation.
- To solve an equation means to find all solutions to the equation.

**Ex.** \[ x^2 = 4 \implies x = \pm 2 \] (two solutions)

**Ex.** solve: \[ 3^x = -1 \]

Notice \( 3^x > 0 \) for any real \( x \).

\[ \rightarrow \text{No solutions.} \]

**Ex.** solve: \[ x + y = 1 \] \( \Rightarrow \) infinite solutions!

i.e. \( x = \frac{1}{2} \) and \( y = \frac{1}{2} \)

\( x = \frac{1}{4} \) and \( y = \frac{3}{4} \)

\( x = 2 \) and \( y = -1 \)
Ex: Solve: \( x^2 = 1.5^x \)

We have no formulas to solve this!

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- Check your solutions by plugging them into the original equation and seeing if it is satisfied.

Ex: Solve \( \log_{10}(x+1) + \log_{10}(x+10) = 1 \)

\[ \Rightarrow \log_{10}(x+1)(x+10) = 1 \]

\[ \Rightarrow (x+1)(x+10) = 10^1 = 10 \]

\[ \Rightarrow x^2 + 11x + 10 = 10 \Rightarrow x^2 + 11x = 0 \]

\[ \Rightarrow x(x+11) = 0 \Rightarrow x = 0 \text{ or } -11 \]

Check \( x = 0 \)

\[ \Rightarrow \log_{10}(0+1) + \log_{10}(0+10) = 1 \]

\[ \Rightarrow 1 + 1 = 2 \checkmark \]

Check \( x = -11 \)

\[ \Rightarrow \log_{10}(-11+1) + \log_{10}(-11+10) = 1 \]

\[ \Rightarrow \log_{10}(-10) + \log_{10}(-1) = 1 \]
But $\log_{10}(-10)$ and $\log_{10}(1)$ are undefined! $\Rightarrow x = -11$ is NOT a solution.