1. Quadratic Equation
2. Completing the Square
3. Quadratic Formula

**Def:** A Quadratic Equation is an equation of the form

$$ax^2 + bx + c = 0$$

where $a$, $b$, and $c$ are real numbers and $X$ is a variable.

**Ex:** $3/4x^2 + \sqrt{2}x - 10 = 0$
Ex: Solve \( x^2 - 7 = 0 \).

\[
\begin{align*}
  x^2 - 7 &= 0 \\
  x^2 &= 7 \\
  x &= \pm \sqrt{7}
\end{align*}
\]

Ex: Solve \( x^2 - 2x + 1 = 0 \).

Assume \( x^2 - 2x + 1 = (x-d)^2 \) for some number \( d \). Then,

\[
  x^2 - 2x + 1 = (x-d)^2 = x^2 - 2dx + d^2
\]

Therefore, \( d = 1 \) and \( 1 = d^2 \).

\[
(x-d)^2 = 0
\]

\[
(x-d) = \sqrt{0} = 0
\]

\[
  x = d
\]
Ex: Solve $x^2 - 2x - 3 = 0$.
Assume $x^2 - 2x - 3 = (x - d)^2$,
then $x^2 - 2x - 3 = x^2 - 2dx + d^2$
and $-2x = -2dx$, $-3 = d^2$
$\Rightarrow 1 = d$ and $\sqrt{-3} = d$
but $1 \neq \pm \sqrt{-3}$!
Our assumption was wrong!
We do know that
$x^2 - 2x + 1 = (x - 1)^2$ and since
$x^2 - 2x - 3 = x^2 - 2x + 1 - 4$
$= (x - 1)^2 - 4$, we can solve
$(x - 1)^2 - 4 = 0$. 
\[ (x-1)^2 = 4 \]
\[ \Rightarrow x-1 = \pm \sqrt{4} = \pm 2 \]
\[ \Rightarrow x = 1 \pm 2 \]

So, \( x = 3 \) or \(-1\).

We just solved the quadratic by completing the square.

**Completing the square**

\[ x^2 + bx = (x + \frac{b}{2})^2 - \left(\frac{b}{2}\right)^2 \]
Ex: Solve \( x^2 + 6x - 4 = 0 \).

**Step 1** Move the constant to the other side.

\[ x^2 + 6x - 4 = 0 \iff x^2 + 6x = 4 \]

**Step 2** Identify \( b \) on the left-hand side

\[ x^2 + bx = x^2 + 6x \]

\[ \Rightarrow b = 6 \]

**Step 3** Transform the left-hand side using the "completing the square" formula, \( x^2 + bx = (x + \frac{b}{2})^2 - \left(\frac{b}{2}\right)^2 \).

\[ x^2 + 6x = (x + \frac{6}{2})^2 - \left(\frac{6}{2}\right)^2 \]

\[ = (x + 3)^2 - 9 \]
Step 4: Solve the resulting equation,

\[ x^2 + 6x - 4 = 0 \iff (x+3)^2 - 9 = 4 \]

\[ \Rightarrow (x+3)^2 = 9 + 4 = 13 \]

\[ \Rightarrow x + 3 = \pm \sqrt{13} \]

\[ \Rightarrow x = -3 \pm \sqrt{13} \]

**Quadratic Formula**

If \( a \neq 0 \) then the solution to \( ax^2 + bx + c = 0 \) is

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
Where did that formula come from?

Answer: Completing the square!

\[ ax^2 + bx + c = 0 \]

\[ x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \]

\[ x^2 + \frac{b}{a}x = -\frac{c}{a} \]

\[ (x + \frac{b}{2a})^2 - \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} \]

\[ (x + \frac{b}{2a})^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2}{4a^2} - \frac{c}{a} \]

\[ (x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2} = \frac{b^2 - 4ac}{4a^2} \]

\[ x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \]
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

**EX**: (If \( b^2 - 4ac > 0 \), then \( ax^2 + bx + c = 0 \) has two real-valued solutions.)

\[ x^2 + 3x + 1 = 0 \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9 - 4}}{2} = \frac{-3 \pm \sqrt{5}}{2} \]

**EX**: (If \( b^2 - 4ac = 0 \), then \( ax^2 + bx + c = 0 \) has only one real-valued solution.)

\[ x^2 + 2x + 1 = 0 \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2}{2} = -1 \]
Ex: If $b^2-4ac < 0$, then $\sqrt{b^2-4ac}$ is NOT a real number, and $ax^2+bx+c=0$ has NO real-valued solutions.

$2x^2 + x + 1 = 0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm \sqrt{7}}{2}$

Not a real number.