\[
\text{Ex: } f(x) = x^2, \ g(x) = 2x + 1 \\
(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2 + 1
\]

\[
\begin{array}{c}
g \circ f(x) \\
g \circ f \\
g \circ f \\
g \circ f \\
\end{array}
\]

(Linear functions act as vertical/horiz. transformations when composed with other functions)

\[
\text{Def: Let } m \text{ and } b \text{ be constants then a linear function is of the form } f(x) = mx + b.
\]
• What is the inverse of a function?
• What functions have an inverse?

\[ f(x) = \frac{x}{x^2 - 1}, \text{ Find } f(z). \]
\[ f(z) = \frac{z}{z^2 - 1} = \frac{z}{3} \]

\[ \text{Ex: } \text{Find } x \text{ such that } f(x) = 2. \]
\[ 2 = f(x) = \frac{x}{x^2 - 1} \Rightarrow 2(x^2 - 1) = x \]
\[ \Rightarrow 2x^2 - x - 2 = 0 \Rightarrow x = \frac{1 \pm \sqrt{1 + 16}}{4} \]
\[ x = \frac{1}{4} \pm \frac{\sqrt{17}}{4} \]

(The inverse question is harder.)
What if there was a function $g$ such that $g(y) = x$ when $f(x) = y$?

\[(g \circ f)(x) = g(f(x)) = g(y) = x\]

**Def:** The identity function, denoted $I(x)$, is defined as

\[I(x) = x\]

**Def:** The inverse of a function, $f$, is denoted $f^{-1}$, and is defined by

\[(f^{-1} \circ f)(x) = x\]
\[(f \circ f^{-1})(y) = y, \quad (i.e. f^{-1} \circ f = I)\]
How can we find the inverse of a function?

- To find the inverse of a function \( f \), solve \( f(x) = y \) to get \( x \) in terms of \( y \).

**Example:** Find the inverse of \( f(x) = 3x + 2 \).

\[
3x + 2 = y \quad \Rightarrow \quad 3x = y - 2
\]
\[
\Rightarrow \quad x = \frac{y - 2}{3} \quad \Rightarrow \quad f^{-1}(y) = \frac{y - 2}{3}.
\]

For what value of \( x \) does \( f(x) = 13 \)?

\[
f^{-1}(13) = \frac{13 - 2}{3} = \frac{11}{3}
\]

Note \( (f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(3x + 2) \)
\[
= \frac{(3x + 2) - 2}{3} = \frac{3x}{3} = x.
\]
Ex: Find the inverse of \( f(x) = x^2 \).
\[ x^2 = y \quad \Rightarrow \quad x = \pm \sqrt{y} \]

Is \( f^{-1}(y) = \pm \sqrt{y} \) a function? (No! It fails the vertical line test.)

What went wrong?

- For a fixed \( y \), \( f^{-1}(y) \) had two values, \( +\sqrt{y} \) and \( -\sqrt{y} \).
- This happened because for a fixed value of \( y \) there are two \( x \) values such that \( f(x) = y \), \( x = +\sqrt{y} \) and \( x = -\sqrt{y} \).
Def: A function $f$ is **one-to-one** if for each number $y$ in the range of $f$, there is exactly one number $x$ in the domain of $f$ such that $f(x) = y$.

**Ex:** $f(x) = x$  **(one-to-one)**

**Ex:** $f(x) = x^2$  **(not one-to-one)**

**Ex:** $f(x) = x^3$  **(one-to-one)**
Horizontal Line Test

The function $f$ is one-to-one if and only if any horizontal line intersects $f$ at most once.

How are the domain and range of a function and its inverse related?

- the domain of $f^{-1}$ equals the range of $f$
- the range of $f^{-1}$ equals the domain of $f$
Let the function \( f(x) = x^2 \). Domain = \([0, \infty)\), then \( f(x) = x^2 \) over \([0, \infty)\) is a one-to-one function. Find its inverse.

\[
\begin{array}{c|c|c}
\text{Domain} & f & f^{-1} \\
\hline
\text{Range} & \[0, \infty\) & \[0, \infty\)
\end{array}
\]

\( y = x^2 \implies \sqrt{y} = x \)

\( \implies f^{-1}(y) = \sqrt{y} \)

over \( y \) in \([0, \infty)\).
Ex: \( f(x) = \sqrt{x} \). Find the domain and range of \( f^{-1} \).

\[
y = \sqrt{x} \quad \Rightarrow \quad y^2 = x \quad \Rightarrow \quad f^{-1}(y) = y^2
\]

<table>
<thead>
<tr>
<th>Domain</th>
<th>( f )</th>
<th>( f^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0, \infty))</td>
<td>([0, \infty))</td>
<td>([0, \infty))</td>
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</table>

Ex: \( f(x) = \frac{1}{x-1} \). Find the domain and range of \( f^{-1} \).

\[
y = \frac{1}{x-1} \quad \Rightarrow \quad x-1 = \frac{1}{y} \quad \Rightarrow \quad x = \frac{1}{y} + 1
\]

\[
\Rightarrow f^{-1}(y) = \frac{1}{y} + 1
\]

<table>
<thead>
<tr>
<th>Domain</th>
<th>( f )</th>
<th>( f^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, 1) \cup (1, \infty))</td>
<td>((-\infty, 0) \cup (0, \infty))</td>
<td>((-\infty, 0) \cup (0, \infty))</td>
</tr>
</tbody>
</table>
Let $f$ be a one-to-one function:

1. $f$ has an inverse
2. If $(a, b)$ is on the graph of $f$, then $(b, a)$ is on the graph of $f^{-1}$.
3. The graph of $f^{-1}$ can be obtained by flipping $f$ across the line through the origin with slope 1.

**Def:** Let $f$ be a function such that if $a, b$ are in the domain of $f$ and $a < b$, then $f(a) < f(b)$, $f$ is an increasing function.
**Def:** Let $f$ be a function such that for any $a, b$ in the domain of $f$ where $a < b$, then $f(a) > f(b)$. Then $f$ is a decreasing function.

**Ex:** $f(x) = x$. Let $a < b$, then $f(a) = a < b = f(b)$, so $f$ is an increasing function.

**Ex:** $f(x) = -x$. Let $a < b$, then $f(a) = -a > -b = f(b)$, so $f$ is a decreasing function.
\[ \exists x: f(x) = x^2 \quad \text{(neither)} \]
\[ \exists x: f(x) = x^3 \quad \text{(increasing)} \]
\[ \exists x: f(x) = -\sqrt{x} \quad \text{(decreasing)} \]

- Increasing and decreasing functions are always one-to-one!

\[ \exists x: f(x) = x^3 \]
\[ \exists x: f(x) = -\sqrt{x} \]