0.2 Algebra

\[ a(x+y) = ax + ay \]

\[ B^0 \quad (a+b)(a+b) = a(a+b) + b(a+b) \]
\[ = a^2 + ab + ba + b^2 \]
\[ = a^2 + 2ab + b^2 \]

\[ \text{ab} = \text{ba} \]

\[ -(\text{a}) = \text{a}, \quad (\text{a})(\text{-b}) = \text{ab}, \quad -\text{a} = (\text{-1})\cdot\text{a} \]

\[ \text{Proofs.} \quad \frac{a}{b} = a \cdot \frac{1}{b} \]
\[ \frac{a}{b} \cdot \frac{c}{a} = \frac{ac}{bd} \quad \frac{a}{b} \cdot \frac{c}{a} = \frac{bc}{ab} = \frac{c}{b} \]
\[ \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + cb}{bd} \]
\[ \frac{a}{b} = \frac{a}{b} \cdot \frac{d}{d} = \frac{ad}{bc} \]

\[ \text{Ex:} \quad \frac{a}{b} = ? \]

\[ \frac{a}{b} = \frac{-a}{-b} = -\frac{a}{b} \quad \frac{-a}{-b} = \frac{(\text{-1})a}{(\text{-1})b} = \frac{a}{b} \]
### 0.3 Inequalities, Intervals and absolute value

- **Real line** (like a ruler)

  - A point on real line represents a real number.
  - Every real number can be found on real line.

  
<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>10/3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- **Negative**
  - Left of zero of real line

- **Positive**
  - Right of zero on the real line

#### Positive and negative numbers

- $\Theta + \Theta = \Theta$
- $\Theta - \Theta = \Theta$
- $\Theta \cdot \Theta = \Theta$

(Additive inverse)

- $\Theta \cdot \Theta = \Theta$
- $\Theta - \Theta = \Theta$

(Additive inverse)

#### Inequality

- $a < b$ means $a$ is shorter than $b$
- $b > a$ means $b$ is taller than $a$

- $a < 5$
- $b > a$ means $b$ is more than $a
We say \( a \) is less than \( b \) if \( a \) is smaller, denote as \( a < b \).

We say \( b \) is greater than \( a \) if \( b \) is bigger, denote as \( b > a \).

\[ a = b \text{ or } b > a \]

\[ a \quad \text{left of } \quad b \quad \text{right of } \quad a \]

\[ a < b \quad b > a \]

Properties:

- **Transitivity:** if \( a < b \), \( b < c \) then \( a < c \)
  
  Similarly: \( a > b \), \( b > c \) then \( a > c \)

- **Addition of inequalities:** if \( a < b \) and \( c < d \) then \( a + c < b + d \)

  Similarly \( a > b \) and \( c < d \) then \( a + c > b + d \)

- **Multiplication:** if \( a < b \), then \( c > 0 \), \( ac < bc \)
  \( c < 0 \), \( ac > bc \) (Ex: \( 1 < 2 \Rightarrow 1 \cdot (-2) > 2 \cdot (-2) \))

- **Additive inverse:** if \( a < b \), \( -a > -b \)
  
  Similarly \( a > b \), \( -a < -b \)

- **Multiplicative inverse:** suppose \( a < b \), if \( a, b \) both positive, \( \frac{1}{a} > \frac{1}{b} \)
  
  If \( a < 0 < b \), \( -\frac{a}{b} < -\frac{1}{a} \)
\[ x + 1 < 2 \]
\[ x + 1 - 1 < 2 - 1 \implies x < 1 \]

\[ x - 8 \over x - 4 \cdot (x - 4) < 3 \cdot (x - 4) \]
\[ \text{Thus, split the case:} \]
\[ \begin{align*}
&\text{if } x > 4 \implies x > x - 4 \implies \\
&\quad \text{then} \\
&\quad \frac{x - 8}{x - 4} < 3 \implies \\
&\quad x - 8 < 3x - 12 \\
&\quad x - 8 - x + 12 < 3x - 12 - x + 12 \implies 4 < 2x \implies x > 2 \end{align*} \]

When \( x > 4 \), the inequality is true if \( x > 2 \), thus \( x > 4 \) is the case.

\[ \begin{align*}
&\text{if } x < 4 \implies x < x - 4 \implies \\
&\quad \text{then} \\
&\quad x - 8 > 3x - 12 \implies x < 2 \\
&\text{Similarly, } x < 4 \text{ is the case.} \\
&\text{Combine the inequalities is true if } x > 4 \text{ or } x < 2 \\
\end{align*} \]

\underline{Intervals (Another representation of inequality)}

Set: a collection of objects (numbers)

\[ \{0, 1\}, \{x : x > 2\}, \{y : y < 0\} \]

Interval: a set of real numbers contains all numbers between 2 numbers, may or may not contain these 2 numbers.
open interval \((a, b) = \{ x : a < x < b \}\)

closed \([a, b] = \{ x : a \leq x \leq b \}\)

half open \((a, b) = \{ x : a < x \leq b \}\)

half open \([a, b) = \{ x : a \leq x < b \}\)

Ex. \((3, 7]\), 3.14 is in \((3, 7]\)

3 is not in \([3, 7]\)

2.9 is not in \((3, 7]\)

7 is in \((3, 7]\)

More Intervals:

\((a, \infty) = \{ x : x > a \}\)

\([a, \infty) = \{ x : x \geq a \}\)

\((\infty, a) = \{ x : x < a \}\)

\((\infty, a] = \{ x : x \leq a \}\)

Ex. \((\infty, 0] : \quad 0 \text{ is in } (\infty, 0]\)

\(-10000\) is in \((\infty, 0]\)

Any negative number is in \((\infty, 0]\)

No positive number is in \((\infty, 0]\)

Union of Intervals (express it as simple as possible)

Ex. \((1, 5) \cup (3, 7] = (1, 7]\)

\(\frac{1}{5} 3 5 7\)
**Absolute Value** (distance from 0)

The absolute value of a number \( b \) is:

\[
|b| = \begin{cases} 
  b & \text{if } b \geq 0 \\
  -b & \text{if } b < 0 
\end{cases}
\]

---

Write \(|x| < 2\) into an interval. W/O. If

- \( x \geq 0 \), then \(|x| = x\) so \( x < 2 \)
- \( x < 0 \), then \(|x| = -x\) so \(-x < 2 \Rightarrow x > -2 \)

Combine \( [0, 2) \cup (-2, 0) \)

\(-2 < x < 2\)

---

Example write \(|x - 5| < 1\) into an interval. W/O abs.

\(-1 < x - 5 < 1\)

\(-1 + 5 < x - 5 + 5 < 1 + 5 \Rightarrow 4 < x < 6 \Rightarrow (4, 6)\)

---

Example write \(|x - 5| > 1\) into an interval. W/O abs.

- If \( x - 5 \geq 1 \)
  \( x - 5 > 1 \Rightarrow x > 6 \)
- If \( x - 5 < 0 \)
  \( x - 5 < -1 \Rightarrow x < 4 \)

\((-\infty, 4) \cup (6, \infty)\)
Functions

input \[ \rightarrow \] function \[ \rightarrow \] output

bread \[ \rightarrow \] toaster \[ \rightarrow \] toasted bread

domain: set of input (specify)
range: set of output

Functions defined by formulas

Example: \( f(x) = x^2 \)

the function, input variable (real number)
can be given as:

\[ f(3) = 3^2 = 9 \]

Substitute \( x \) to 3

\[ f(1+\alpha) = (1+\alpha)^2 = \alpha^2 + 2\alpha + 1 \]

\[ f\left(\frac{x+5}{2}\right) = \left(\frac{x+5}{2}\right)^2 = \frac{x^2+10x+25}{4} \]

Example:

\[ f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \]

\[ f(1+5) = \begin{cases} x+5 & \text{if } x+5 \geq 0 \\ -(x+5) & \text{if } x+5 < 0 \end{cases} \]

\[ f(-1) = 1 \quad f(1) = 1 \]
Input Variable can be anything

\[
\begin{align*}
\text{f(x)} &= x^2 \\
\text{f(t)} &= t^2 \\
\text{f(\square)} &= \square^2
\end{align*}
\]

does not matter what letter you are using.

things matter is that you know it's an input variable

can be changed, substituted by other letter or number

---

Range and Domain

Domain: where function defined,
set of input real numbers
* if not specified, the set of all real number with
for which the formula makes sense. (valid)

Ex. \( f(x) = x^2 \)

Domain: \([1, 2]\)

\( f(0) \) is not defined since \( 0 \) is not in \([1, 2]\)

Ex. \( f(x) = \frac{1}{x} \)

What is the domain \((x \neq 0)\)

\((-\infty, 0) \cup (0, +\infty)\)

Range: set of output, all \( y \) s.t. \( f(x) = y \) for at least one \( x \) in range.

Ex. \( f(x) = x \)

Domain: \([1, 2]\)

Range is \([1, 2]\)

Ex. \( f(x) = x^2 \)

Domain: \([-1, 1]\)

Range is \([0, 1]\)
* Important about functions:

For every input, there is only one output.

However, one output may have more than one input.

Ex: \( f(x) = x^2 \)  \( f(-1) = 1 \)  \( f(1) = 1 \)

**Function as a table:**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>7</td>
<td>49</td>
</tr>
<tr>
<td>13</td>
<td>169</td>
</tr>
</tbody>
</table>

**Domain:** \( \{2, 3, 7, 13\} \)

**Range:** \( \{4, 9, 49, 169\} \)

**Equality of functions:**

Two functions are equal if and only if they have the same domain and the same value at every number in that domain.

For: \( f(x) = x^2 \)  \( g(x) = x^2 \)

\( f(3) = 9 \)  \( g(3) = 9 \)

\( f(1) = 1 \)  \( g(1) = 1 \)

\( f(15) = 225 \)  \( g(15) = 225 \)
§ 1.2 Coordinate Plane & Graph

**Chess Board**

```
A B C
1 2 3
```

**Coordinate plane (Cartesian plane)**

- Vertical axis (up pos)
- Horizontal axis (right pos)

**Intersection is the origin**

**Elements of Coordinate plane**

- **Horizontal axis** (real line) right pos. with 0,1
- **Vertical axis** (real line) up pos. with 0,1
  - which S has same zero
- **Point of Origin** = Intersection of (H/V axis)

**Coordinate and point on plane**

- Distance to the Horizontal axis
- Distance to the Vertical axis

```
(1,2) 3 2 1
-2 -1 0 1 2 3
```

**Graph of function**

**Def.** The graph of function is the set of points of the form (x, f(x))

**Ex.**

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>
Ex: \( f(x) = |x| \) (Domain is real line)

- cannot show the whole graph but a part of them

Ex (How to graph \( f(x) \))

\[ f(x) = x^2 \] Domain \([0, 2]\)

1. Pick few points, find them on the plane
2. Connect them smoothly

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1/2</td>
<td>1/4</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3/2</td>
<td>9/4</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

What can you tell from graph

1. Find the values of a function from its graph

Ex: \( f(\frac{3}{2}) = \frac{3}{2} \)

Evaluate \( f(\frac{3}{2}) \)

- find \( x = \frac{3}{2} \) on horizontal axis
- draw the vertical line pass the pt.
- find the intersection with the graph
- draw the horizontal line pass the intersection
- the horizontal exceeds on vertical axis

\[ f(\frac{3}{2}) \]

2. Determine Domain and Range

Assume your fun is given by graph.

\( a \) is in the Domain if \( x = b \) intersects with the graph of the fun.

\( b \) is in the Range if \( y = a \) intersects with the fun.
By 4

\[ y = x^2 \]

Is 1.5 m range? Yes
Is 5 m range? No
Range is? [0, 4]

3. Determine if the graph is a function
Vertical line test
if \( x = 5 \) only intersects with it at most once

Yes
No.
Section 1.3: Function Transformations and Its Graphs

Vertical transformation: Shifting, Stretching, Flipping
* Shifting: graph for \( y = f(x) \) \( \Rightarrow \) what is the graph of \( y = f(x) + k \)

- \( f(x) = x^2 \), Domain \([1, 1]\) \( \Rightarrow \) \( f(x+1) = (x+1)^2 \)
- \( f(x) = x^2 - 1 \), Domain \([-1, 1]\) \( \Rightarrow \) \( f(x-1) = (x-1)^2 \)

The graph \( y = f(x) + a \) is obtained by shifting \( y = f(x) \) up \( a \) units
- \( y = f(x) - e \) is obtained by shifting \( y = f(x) \) down \( e \) units

* Stretching given for \( f(x) = f(x) \) \( \Rightarrow \) what is the graph of \( y = cf(x) \) \( (c > 0) \)

- \( f(x) = x^2 \), Domain \([1, 1]\) \( \Rightarrow \) \( g(x) = 2x^2 \), Domain \([-1, 1]\)
- \( f(x) = \frac{1}{2} x^2 \)

The graph of \( y = cf(x) \) is obtained by vertically stretching \( f \) by a factor of \( c \).
The graph of $g(x) = -f(x)$ is obtained by flipping $f$ across the horizontal axis.