§ 2.1 Linear functions

Graphical: straight lines (except vertical line)

Formulas:
\[ y = \frac{y_2 - y_1}{x_2 - x_1} \]
\[ y = \frac{y_4 - y_3}{x_4 - x_3} \]

Slope:
\[ \text{slope} = \frac{\text{rate of change}}{\text{rate of change}} \]

Definition:
\[ \text{for any } x_1, x_2, x_3, x_4, \text{ where } x_1 \neq x_2, x_3 \neq x_4 \]

Definition:
\[ \text{for any two pts on a line, slope of the line is:} \]
\[ \frac{y_2 - y_1}{x_2 - x_1} \]

The equation of the line

Point + slope: given \((x_1, y_1)\) on the line, \(m\) slope, \(m\)

\[ \frac{y - y_1}{x - x_1} = m \Rightarrow y - y_1 = m(x - x_1) \]

Example:
Slope \(\frac{1}{4}\), contains \((4,1)\)

\[ y - 1 = \frac{1}{4}(x - 4) = \frac{1}{4}x - 1 \]
\[ \Rightarrow y = \frac{1}{4}x \]
* slope + y-intercept: \( m, b \)
\[
y = mx + b
\]
be the line passes \((0, b)\)
points on y-axis

* 2 points : \((x_1, y_1), (x_2, y_2)\) (best way to sketch a line)

\[
slope = \frac{y_2 - y_1}{x_2 - x_1}
\]

Equation: \( y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \)

Ex: points \((2, 4), (5, 1)\)

\[
slope = \frac{1 - 4}{5 - 3} = -1
\]

\[
y - 4 = -1(x - 2) = -x + 2 \Rightarrow y = -x + 6
\]

Ex: Find the function which takes temperature in °C as input and °F as output. Given \( f(0) = 32 \) \(, f(100) = 212 \)

\[
f(0) = 32 \Rightarrow b = 32 \Rightarrow f(x) = mx + 32
\]

put \( f(100) = 212 \Rightarrow m\cdot 100 + 32 = 212 \Rightarrow m = \frac{9}{5}
\]

\[
f(x) = \frac{9}{5}x + 32
\]

Constant function: \( f(x) = b \), \( b \) is constant, \( m = 0 \) for horizontal
**Parallel**

**Definition**

Two lines are parallel if and only if they have the **same slope**.

**Graphical**

Two lines never intersect.

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**Exercise**

Find the equation of the line that contains the point (1,1) and is parallel to the line containing (2,2) and (4,1).

Assume the slope of the line is $m$.

Given $y-1 = m(x-1)$, to find $m$, we know $m$ is same as the slope of the line containing (2,2) and (4,1), so the slope of

Thus \[ m = \frac{1-2}{4-2} = -\frac{1}{2}. \]

Thus the equation is

\[ y-1 = -\frac{1}{2}(x-1) \Rightarrow y = -\frac{1}{2}x + \frac{3}{2}. \]

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**Intersection of 2 lines**

**Diagram**

- Intersect: a point, 2 lines pass it.
- $y = x$
- $y = x+2$

Thus $x_0 = y_0$ and $-x_0 + 2 = y_0$

\[ \Rightarrow x_0 = y_0 = -x_0 + 2 \Rightarrow x_0 = 1 \quad y_0 = 1 \]

**Intersection**: (1,1)

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**Exercise**

Find the intersection of $y = M_1x + b_1$, $y = M_2x + b_2$.

$m_1x + b_1 = M_2x + b_2$

\[ \Rightarrow (M_1 - M_2)x = b_2 - b_1 \]

If $M_1 = M_2$ and $b_2 \neq b_1$, **No solution** (parallel).

If $M_1 = M_2$ and $b_2 = b_1$, **Same line**.

If $M_1 \neq M_2$,

\[ x = \frac{b_2 - b_1}{M_1 - M_2}, \quad y = \frac{M_1b_2 - M_2b_1}{M_1 - M_2} \]

**Intersection**: \( \left( \frac{b_2 - b_1}{M_1 - M_2}, \frac{M_1b_2 - M_2b_1}{M_1 - M_2} \right) \)
Two lines are perpendicular if and only if the product of their slopes is -1.

Ex: Show \( y = x \) is perpendicular to \( y = -x + 2 \)

Slope of \( y = x \) is 1. Slope of \( y = -x + 2 \) is -1.

\[ 1 \cdot (-1) = -1, \] thus by def. they are perpendicular.

Ex: Find \( t \), s.t. the line containing \((1, -2)\) and \((3, 3)\) is perpendicular to the line containing \((9, -1), \,(t, 1)\).

Slope 1: \( \frac{3 - (-2)}{3 - 1} = \frac{5}{2} \) 
Slope 2: \( \frac{t - (-1)}{t - 9} = \frac{2}{t - 9} \)

\[ \frac{5}{2} \cdot \frac{2}{t - 9} = -1 \Rightarrow \frac{5}{t - 9} = -1 \Rightarrow 5 = -t + 9 \Rightarrow t = 4 \]

Equation and function (set of points)

- Equation \( xy = k \) can describe a curve in coordinate plane, the curve can be function or not function. \((x = 4)\)
- Function \( y = f(x) \) \( \iff \) graph of \( f \) is vertical line test
Completing the square

\[(x+a)^2 = x^2 + 2ax + a^2\]

By

\[x^2 + 6x + 4 = (x + 3)^2 - 9 + 4\]

\[= (x + 3)^2 - 5\]

**Formula**

\[x^2 + bx = (x + \frac{b}{2})^2 - \frac{b^2}{4}\]

\[b\] can be negative

**Ex**

\[x^2 - 10x\]

\[b = -10, \quad \frac{b}{2} = -5, \quad \frac{b^2}{4} = 25\]

**General formula**

\[a(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a}\]

**Proof**

\[a(x + \frac{b}{2a})^2 + c = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c = a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a}\]

Solve quadratic equation

2. Solve \(x^2 + 6x - 4 = 0\)

1. Complete the square:

\[(x + 3)^2 - 9 + 4 = 0\]

2. \((x + 3)^2 = 13\)

3. \(x + 3 = \pm \sqrt{13} \Rightarrow x = -3 \pm \sqrt{13}\)

Solve \(ax^2 + 5x + c = 0\)

\[a(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2} \Rightarrow (x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}\]

**Take root? discuss!**

- If \(b^2 - 4ac > 0\) then \(x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\)
- If \(b^2 - 4ac = 0\) then \(x = -\frac{b}{2a}\)
- If \(b^2 - 4ac < 0\) No solution!
Find \( s \) and \( t \) such that \( s + t = 8 \) \( s - t = 12 \)

\[
\begin{align*}
S + t &= 8 \quad \Rightarrow \quad S = 8 - t \\
S - t &= 12 \quad \Rightarrow \quad (8 - t) + t = 12 \quad \Rightarrow \quad \frac{t^2 - 8 + 12}{a} = 0 \\
&= \frac{4}{a} \\
b^2 - 4ac &= 8^2 - 4 \cdot 1 \cdot 12 = 16 > 0
\end{align*}
\]

Thus \( t = \frac{-8 \pm \sqrt{16}}{2(1)} = 2 \) or \( 6 \)

\[
\begin{align*}
\text{If } t &= 2 \quad \Rightarrow \quad S = 6 \\
\text{If } t &= 6 \quad \Rightarrow \quad S = 2
\end{align*}
\]

Quadratic functions and parabola

Quadratic function \( f(x) = ax^2 + bx + c \)

\[
\begin{align*}
f(x) &= x^2, \quad a \neq 1, \quad b = 0, \quad c = 0
\end{align*}
\]

The graph of quadratic function is parabola

Vertex of parabola is the pt where the line of symmetry intersects the parabola

\[
y = x^2 \quad \text{line of symmetry } x = 0 \\
\text{Vertex } (0, 0)
\]

\[
y = (x + h)^2 = f(x) \quad \text{if } f(x) = x^2 \quad f(x) = g(x + h)
\]

\[
y = (x - h)^2 = f(x) \quad \text{if } f(x) = x^2 \quad f(x) = g(x - h)
\]

Line of symmetry \( x = -1 \)

Line of symmetry \( x = 1 \)

Line of symmetry \( x = 1 \)
Quadratic function, another

\[
\begin{align*}
\text{for } ax^2 + bx + c & \\
&= a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a} \\
&= a(x - h)^2 + k \\
h &= -\frac{b}{2a} \quad k &= -\frac{b^2 - 4ac}{4a}
\end{align*}
\]

Notice: for any real number \( z \), \( z^2 \geq 0 \)

So: \((x-h)^2 \geq 0 \)

If \( a > 0 \), \( a(x-h)^2 + k > 0 \) \( \Rightarrow \) \( a(x-h)^2 + k \geq k \)

If \( a < 0 \), \( a(x-h)^2 \leq 0 \) \( \Rightarrow \) \( a(x-h)^2 + k \leq k \)

Suppose \( f \) is a quadratic function, with form:

\[
f(x) = a(x-h)^2 + k
\]

* \( h \) line of symmetry: \( x = h \)

* Vertex: \((h, k)\) or \((h, f(h))\)

* If \( a > 0 \) then \( f \) has min value \( k \), when \( x = h \)

* If \( a < 0 \) then \( f \) has max value \( k \), when \( x = h \)

Circle

Put a circle into the coordinate plane

What is the equation of the circle?

Distance (Pythagorean Thm.)

\[
\begin{align*}
\sqrt{3^2 + 4^2} &= 5 \\
\end{align*}
\]

distance on hor. line

distance on ver. line
Max General

\[ r^2 = 3^2 = 5 = \sqrt{(5-1)^2 + (4-1)^2} < \text{distance between (5,1) \& (1,4)} \]

Distance \((x_1, y_1) \& (x_2, y_2)\) is \(\frac{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}}{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}}\)

Circle

Put center at (2,0)

All pts on the circle satisfies distance between pt on circle and center is a constant r

Distance between pt on circle and center is a constant r

Equation of a circle:

The circle with center \((h,k)\) and radius \(r\) has equation:

\((x-h)^2 + (y-k)^2 = r^2\)

Find the radius and center of the circle in the coordinate plane:

\(x^2 + 4x + y^2 - 6y = 12\)

\[ (x+2)^2 - 4 + (y-3)^2 - 9 = 12 \]

\[ (x+2)^2 + (y-3)^2 = 25 \]

Center \((-2, 3)\)

Radius \(\sqrt{25} = 5\)