The Ricci curvature is a tensorial quantity in differential geometry and the study of smooth manifolds. This tensor is actually a tensor field, so restricting it to a single point produces a tensor known as an Algebraic Curvature Tensor (ACT). We study possible ACTs in order to gain insight about the overall tensor field. If \( V \) is a real vector space of finite dimension \( n \) then \( R = \otimes^n V^\ast \) is an Algebraic Curvature Tensor if:

\[
R(x, y, z, w) = R(y, z, w, x) = R(z, w, x, y) = R(w, x, y, z) = 0
\]

for all \( x, y, z, w \in V \). The following class of ACTs is particularly important. Take \( x, y, z, w \in V \) and some symmetric bilinear form \( \phi \in S^2(V^\ast) \) and define

\[
R_{\phi}(x, y, z, w) = \phi(x, z)R(y, w) - \phi(z, w)R(x, y)
\]

These are known as the canonical Algebraic Curvature Tensors. Defining \( A|\phi \) as the set of all possible ACTs on \( V \), it is known \([3]\) that

\[
A|\phi = \text{span} \{R_{\phi}[\omega] \in S^4(V^\ast)\}
\]

A similar definition and result exist for anti-symmetric forms, but this is not explicitly relevant to our results.

**Goals and Motivation**

In particular, we will be studying the structure groups of these objects. The structure group of some covariant tensor, or group of covariant tensors, is defined as the group of endomorphisms on the vector space \( V \) which preserve the tensors under precomposition. For example, the structure group of a weak model space \( W \), using the notation \( A|W \), is defined as \( A|W = \{A : \text{Aut}(A|W) \} \). We consider the structure groups of several broad classes of model spaces, and explore the connections between the structure of \( G_M \) and the structure of \( M \). In particular, we use \( R = R_{\phi} \) and \( R = R_{\phi} \otimes R_{\phi} \) for \( \phi \in S^2(V^\ast) \) and we also study structure groups of \( \phi \in S^4(V^\ast) \).

It is often desired to know if a given manifold is locally homogeneous. It is also of interest to determine when manifolds are \( k \)-curvature homogeneous, which means that the first \( k \) covariant derivatives of \( R \) are each locally constant. There exist what are known as weak \( k \)-curvature invariants, built by contractions of \( R \), and in the Riemannian these scalar functions are all constant if and only if the manifold is locally homogeneous \([4]\). This result can not be generalized to the pseudo-Riemannian case. In fact, there exist vanishing scalar invariants \( \mathcal{S} \)-manifolds for which all scalar contractions of \( R \) are zero, but not all of these manifolds are locally homogeneous. However, there are classes of pseudo-Riemannian manifolds for which alternate scalar invariants have been found (for an example, see \([1]\)). Thus, by studying the structure group of a given model space, we attempt to form new invariants which will also work in the pseudo-Riemannian case.

**Structure Groups of \( R_{\phi} \)**

First consider the simplest case, for some \( M = (V, \langle \cdot, \cdot \rangle, R) \) let \( R: R_{\phi} \). It is known that this \( R \) corresponds exactly to the case of constant sectional curvature \([2]\). We give our own proof to the following known result.

**Lemma 1**. For the model space \( M = (V, \langle \cdot, \cdot \rangle, R) \) (with the inner product positive definite) we have:

\[
G_M = G_{R_{\phi}} \equiv R = E_{R_{\phi}} \equiv R \text{ has constant sectional curvature, } k.
\]

**Outline of Proof**. Assume \( R \) has constant \( \kappa \). Clearly \( G_{R_{\phi}} \subset G_M \) since \( R = R_{\phi} \) and by definition we have the opposite inclusion, so one implication is complete. Now we must assume \( G_M = G_{R_{\phi}} \) and show \( R \) has constant \( \kappa \). In order to do this we consider a 2-plane and its image under \( k \in G_M \), which is also a 2-plane. Using the definition of sectional curvature we are able to show that those two arbitrary planes must have the same \( \kappa \).

We also consider more general cases. It is almost always true that \( G_{R_{\phi}} = G_R \) on the weak model space with \( \phi \in S^2(V^\ast) \), but in the balanced signature case this does not necessarily hold.

**Lemma 2**. Let \( \phi \in S^2(V^\ast) \) with rank greater than or equal to 2. Then the following are true:

1. If \( A \in G_R \), then \( A^\phi \equiv \pm A \).
2. If the signature is unbalanced then \( G_R = G_{R_{\phi}} \).

**Outline of Proof**. We know \( A \in G_{R_{\phi}} \), so \( A^\phi \equiv \pm A \). This is a better in \( G_R \) or \( \sigma \)-isometry of \( \phi \), which can only exist in the balanced signature case.

**The Elements of \( G_{\mathcal{W}} \)**

We begin by exploring the connection between the decomposition of the model space and the structure of \( G_{\mathcal{W}} \).

**Lemma 3**. Assume \( M = (V, \langle \cdot, \cdot \rangle, R) \) with \( \mathcal{W} \) as the structure group for \( V, R_1, R_2 \). Then \( V \), \( R_1 \), and \( R_2 \) are invariants spaces for all \( A \in G \equiv G_{\mathcal{W}} \)

This allows us to relate the irreducible subspaces of \( G_{\mathcal{W}} \) viewed from a representation theory point of view to the indecomposable subspaces of \( M \) viewed from a linear algebraic point of view. Another result describes an important decomposition of a model space for when the kernel of \( R \) is non-trivial.

**Lemma 4**. Define \( \mathcal{T} = \ker R \) and also define \( \mathcal{T} = (V, \langle \cdot, \cdot \rangle) \) to be a projection. If \( \mathcal{T} \) is defined by \( \pi \mathcal{T} = R \) as an algebraic curvature tensor on \( V, \langle \cdot, \cdot \rangle \) then \( G \equiv G_{\mathcal{T}} \).

Using this decomposition we can begin to see the form of elements of \( G_{\mathcal{W}} \). Notice that the vector spaces are not invariant spaces in the following sense, because \( \ker R \not\equiv \{0\} \).

**Theorem 1**. On some \( V \) of dimension \( n \) with \( \phi \in S^2(V^\ast) \) of rank \( k \) that for any \( A \in G_{\mathcal{W}} \) written as a matrix in the orthonormal basis ordered so that the null vectors are last, we have

\[
A = \mathcal{T} \oplus \mathcal{W}
\]

where \( \mathcal{T} \subset \mathcal{W} \). It can be any \( (n-k) \times k \) matrix, and \( C \subset \mathcal{W} \). \( \mathcal{W} \subset \mathcal{W} \).

**Outline of Proof**. Clearly \( \mathcal{T} \subset \mathcal{W} \) because it only acts on \( \mathcal{T} \) and can not even know about the other parts of \( V \). The zero block must be there since the \( \mathcal{T} \) is invariant. The \( \mathcal{W} \) block can be anything because this block is simply sending portions of vectors to the kernel, which does not effect anything. Finally, the \( \mathcal{C} \) block has to be full rank only to keep \( A \in G_{\mathcal{W}} \).

**Conclusion and Opportunities for Further Research**

- Notice that by combining the results of Theorem 1 and Theorem 2 we have a lot of information describing the elements of a given structure group.
- We strongly believe that Lemma 1 can be generalized to include all non-degenerate bilinear forms without much change to the proof.
- This project was initiated in order to find new invariants for pseudo-Riemannian model spaces, so making the connection back to that original goal would be important.
- Theorem 2 can certainly be written entirely from the viewpoint of representation theory, and it would be valuable to make this translation and consider its ramifications in that subject.
- Considering the special case of \((V, \langle \cdot, \cdot \rangle) = (V, R_{\phi}) \) with each \( \phi \in S^2(V^\ast) \) could potentially lead to interesting results that would depend on the relation between \( \phi \) and \( \psi \).

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**References**