Asymptotics for Empirical Process and Bootstrap

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Overview

Reference

- Aad van der Vaart, *Asymptotic Statistics*, Ch. 19 and Ch. 23. Cambridge University Press, 1998
- Aad van der Vaart, Jon Wellner, *Weak Convergence and Empirical Processes*. Springer, 1996
- Evarist Giné, Joel Zinn, *Some Limit Theorems for Empirical Processes*. The Annals of Probablity Vol. 12, No. 4, 1984
- Evarist Giné, Joel Zinn, *Necessary Conditions for the Bootstrap of the Mean*. The Annals of Statistics Vol. 17, No. 2, 1989
- Evarist Giné, Joel Zinn, *Bootstrapping General Empirical Measures*. The Annals of Probability Vol. 18, No. 2, 1990

Empirical Measure and Bootstrap Measure

- Empirical cumulative distribution function:

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} \chi_{[X_i,+\infty)}(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x)$$

- Empirical measure:

$$P_n(\omega) = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i(\omega)}, \omega \in (\Omega^\infty, \mathcal{P}^\infty, P^\infty)$$

- Bootstrap measure:

$$P_n^*(\omega, \sigma) = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i^*(\omega,\sigma)} = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_\sigma(\omega)}$$

$\sigma \sim$ Multinomial(n) with uniform p_i

Glivenko-Cantelli Theorem on $\mathbb{R}$

Theorem (Glivenko-Cantelli)

$$\|F_n - F\|_\infty \xrightarrow{a.s.} 0.$$

Proof by partition, pick bigger jumps of $F(x)$ as cut points.

Càdlàg space and Donsker Theorem

Càdlàg space $D[-\infty, +\infty]$, right continuous functions with left limits.
Skorokhod metric:

$$\sigma(f, g) = \inf_{\lambda \in \Lambda} \max \|\lambda - I\|, \|f - g \circ \lambda\|$$

Λ is the set of all strictly increasing continuous bijection of $[-\infty, +\infty]$.

Theorem (Donsker)

In Skorokhod topology of Càdlàg space $D[-\infty, +\infty]$,

$$\sqrt{n}(F_n - F) \xrightarrow{\mathcal{L}} B \circ F$$

where B is a Brownian bridge.

Weak Convergence in $l^\infty(\mathbb{R})$

Fact:

- F_n and $G_n = \sqrt{n}(F_n - F)$ are not Borel measurable $(\mathcal{P}^n \to \mathcal{B}(l^\infty(\mathbb{R})))$.
- $l^\infty(\mathbb{R})$ is neither compact nor separable.

Thus, Dudley and Hoffman-Jørgensen developed the extended theory of weak convergence.

Definition (Outer expectation)

$\mathbb{E}^* T(P) = \inf\{\mathbb{E}U : U \geq T, U\text{extended r.v and } \mathbb{E}U = \int UdP \text{ exists}\}$

Definition (Weak Convergence)

$G_n \to G$ in $l^\infty[0,1]$. For all bounded continuous $h : l^\infty[0,1] \to \mathbb{R}$,

$$\mathbb{E}^* h(G_n) =\to \mathbb{E}h(G)$$

Second Donsker Theorem

Theorem (Donsker)

If F is continuous, then G_n converges weakly in $l^{\infty}(\mathbb{R})$ to $B \circ F$, a tight process concentrating on a complete separable subspace of $l^{\infty}(\mathbb{R})$.

Empirical Process in General Sample Space

- No more c.d.f. $F_n(.)$ and $F(.)$, all in terms of measure P_n and P
- For a measurable function $f : \Omega \to \mathbb{R}$,

$$P_n f = \frac{1}{n} \sum_{i=1} n f(X_i), \qquad Pf = \int f dP$$

- No proper extension to Càdlàg and Skorokhod, but $l^\infty(\mathcal{F})$, where $\mathcal{F}$ is a class of functions.

P-Glivenko-Cantelli and P-Donsker

Suppose $\mathcal{F}$ is a class of measurable functions.

Definition (P-Glivenko-Cantelli)

$$\|P_n f - Pf\|_{\mathcal{F}} = \sup_{f \in \mathcal{F}} |P_n f - Pf| \xrightarrow{a.s.} 0.$$

Definition (P-Donsker)

$G_n = \sqrt{n}(P_n - P)$ converges in law to a tight limit process G_P in $l^\infty(\mathcal{F})$, also known as a P-Brownian bridge.

In Giné and Zinn (1984), there is a long list of criteria for proper class $\mathcal{F}$. Usually, we need additional measurability for uncountable $\mathcal{F}$:

- LSM
- SM
- LDM
- DM
- NLSM
- NLDM

Empirical Bootstrap

In Giné and Zinn (1990), a general convergence theorem for empirical Bootstrap is established. We need to assume certain measurability condition $\mathcal{F} \in M(P)$ NLDM(P) for $\mathcal{F}$ and NLSM(P) for $\mathcal{F}^2$ and $\mathcal{F}'^2$.

Theorem (Giné and Zinn 1990)

Let $\mathcal{F} \in M(P)$, then the following are equivalent:

(a) *The envelope F for $\mathcal{F}$ is in $L^2(P)$ and $\mathcal{F}$ is P-Donsker with limit G_P.*

(b) *There exists a centered tight Gaussian process G on $\mathcal{F}$ such that $\sqrt{n}(P_n^* - P_n) \to G$ weakly in $l^\infty(\mathcal{F})$.*

If either one holds, then $G = G_P$.

Convergence via Bounded Lipschitz Metric

The equivalence of weak convergence in $l^\infty(\mathcal{F})$:

$$\mathcal{L}\{G_n\} \rightsquigarrow \mathcal{L}\{G\} \Leftrightarrow \sup_{h \in BL_1(l^\infty(\mathcal{F}))} |\mathbb{E}^* h(G_n) - \mathbb{E} h(G)| \to 0$$

where BL_1 is the space of functions whose Lipschitz norm is bounded by 1.

Theorem

For every P-Donsker class $\mathcal{F}$ with envelope function F, i.e. $|f(\omega)| \leq F(\omega) < \infty$ for all $\omega \in \Omega$ and $f \in \mathcal{F}$.

$$\sup_{h \in BL_1(l^\infty(\mathcal{F}))} |\mathbb{E}_M h(G_n^*) - \mathbb{E} h(G_P)| \xrightarrow{P} 0$$

Moreover, G_n^* is asymptotically measurable. If $P^* F^2 < \infty$, then the convergence is outer almost surely as well.

Theorem (Delta method for Bootstrap)

Let $\mathbb{D}$ be a normed space and let $\phi : \mathbb{D}_\phi \subset \mathbb{D} \to \mathbb{R}^k$ be Hadamard differentiable at θ tangentially to a subspace $\mathbb{D}_0$. Let $\hat{\theta}_n$ and $\hat{\theta}^*$ be maps with values in $\mathbb{D}_\phi$ such that

- $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} T$, *tight in* $\mathbb{D}_0$.
- $\sup_{h \in BL_1(\mathbb{D})} |\mathbb{E}_M h(\sqrt{n}(\hat{\theta}_n^* - \hat{\theta})) - \mathbb{E}h(T)| \xrightarrow{P} 0.$

Then $\sup_{h \in BL_1(\mathbb{D})} |\mathbb{E}_M h(\sqrt{n}(\phi(\hat{\theta}_n^*) - \phi(\hat{\theta}))) - \mathbb{E}h(\phi'_\theta(T))| \xrightarrow{P} 0.$

An Application

Corollary (Empirical distribution function)

The class $\mathcal{F} = \{f_t : f_t = 1_{(-\infty,t]}\}$ is Donsker, so the empirical distribution function F_n satisfies the condition for the preceding theorem. Thus, conditionally on sample, $\sqrt{n}(\phi(F_n^*) - \phi(F_n))$ converges in distribution to the same limit as $\sqrt{n}(\phi(F_n) - \phi(F))$, for every Hadamard-differentiable function ϕ, e.g. quantiles and trimmed-means.

The End