Math 181A Homework 10 Solution

7.4.9

(a)

\( \bar{y} = 35.42, \; S = 7.23. \)

By Theorem 7.4.1,

\[
CI = (35.42 - t_{0.025,11} \frac{7.23}{\sqrt{12}}, \; 35.42 + t_{0.025,11} \frac{7.23}{\sqrt{12}}) = (30.82, 40.01)
\]

(b) There is no clear pattern from the plot. May assume randomness.

\[
\begin{array}{c}
\text{Scientists: Age of Prime} \\
\text{Age} \\
\text{Year} \\
1500, 1700, 1800, 1900
\end{array}
\]

7.4.19

\( y_1, \ldots, y_n \sim iid \mathcal{N}(\mu, \sigma^2). \; \bar{y} = 37.06, \; S = 5.05 \)

A proper hypothesis is

\[
H_0 : \mu = 40 \text{ versus } H_1 : \mu < 40
\]

\[
t = \frac{-2.25}{1.76} = -t_{0.05,14}
\]

\( H_0 \) is rejected.
7.4.25
The short answer is \( t_n \xrightarrow{} \mathcal{N}(0,1) \). Proof needs advanced results including Slutsky.

7.5.3
(a) 2.09
(b) 7.26
(c) \[ \mathbb{P}(\chi^2_{22} < 9.542) = 0.01 \Rightarrow \mathbb{P}(\chi^2_{22} \leq y) = 0.1 \Rightarrow y = 14.04 \]
(d) \[ \mathbb{P}(\chi^2_{31} \leq 48.232) = 0.975 \Rightarrow \mathbb{P}(\chi^2_{31} < y) = 0.025 \Rightarrow y = 17.54 \]

7.5.5
Forget the table. This is 21st century.
\[ \mathbb{P}(\chi^2_{200} \leq 234) = 0.95 \]

7.5.8
\[ \frac{18S^2}{12} \sim \chi^2_{18} \Rightarrow \mathbb{P}\left(\frac{2}{3}\chi^2_{0.025,18} \leq S^2 \leq \frac{2}{3}\chi^2_{0.975,18}\right) = 0.95 \]
\[ a = 5.49, b = 21.02 \]

7.5.15
\[ \chi^2 = \frac{18 \times 733.4}{30.4^2} = 14.28 \geq 9.39 = \chi^2_{0.05,18} \]
We cannot reject \( H_0 \).

Additional Problem:
\( X_1, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda) \), \( \alpha = 0.05 \)

\( H_0 : \lambda = 2 \) versus \( H_1 : \lambda \neq 2 \).

The likelihood function:

\[
L(\lambda) = \frac{\lambda^{\sum_{i=1}^{n} X_i}}{\prod_{i=1}^{n} X_i !} e^{-n\lambda}
\]

Log-likelihood:

\[
l(\lambda) = \sum_{i=1}^{n} X_i \log(\lambda) - n\lambda + \text{something irrelevant}
\]

1st derivative:

\[
\frac{\partial l(\lambda)}{\partial \lambda} = \frac{\sum_{i=1}^{n} X_i}{\lambda} - n
\]

Critical point: \( \hat{\lambda} = \bar{X} \).

2nd derivative:

\[
\frac{\partial^2 l(\lambda)}{\partial \lambda^2} = -\frac{\sum_{i=1}^{n} X_i}{\lambda^2} < 0
\]

Convex, thus MLE is \( \hat{\lambda} = \bar{X} \).

Fisher Information:

\[
I(\lambda) = \mathbb{E} \frac{X}{\lambda^2} = \frac{1}{\lambda}
\]

Under \( H_0 : \lambda = 2 \),

\[
\sqrt{nI(2)(\hat{\lambda} - 2)} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1)
\]

Let \( Z = \sqrt{n/2}(\bar{X} - 2) \), reject when \( |Z| \geq 1.96 \).