Math 181A Homework 4 Solution

5.3.10

$K \sim Binom(540, p)$, so apply the Theorem 5.3.1

$$CI = \left[ \frac{192}{540} - 1.96 \sqrt{\frac{\frac{192}{540} \left( 1 - \frac{192}{540} \right)}{540}}, \frac{192}{540} - 1.96 \sqrt{\frac{\frac{192}{540} \left( 1 - \frac{192}{540} \right)}{540}} \right] = [0.315, 0.396]$$

5.3.24

See Example 5.3.3. And the About the Data following.

5.4.10

By the hint, let us check $\hat{\theta}^2 = Y^2$ first.

$$E\hat{\theta}^2 = \int_0^\theta y^2 \frac{1}{\theta} dy = \frac{1}{3} \theta^2$$

Then, construct $\bar{\theta}^2 = 3\hat{\theta}^2 = 3Y^2$

$$E\bar{\theta}^2 = 3E\hat{\theta}^2 = 3 \frac{1}{3} \theta^2 = \theta$$

Indeed, it is unbiased.

5.4.12

$$E\hat{\sigma}^2 = E \frac{1}{n} \sum_{i=1}^n (Y_i - \mu)^2$$

$$= \frac{1}{n} \sum_{i=1}^n E(Y_i - \mu)^2$$

$$= \frac{1}{n} \sum_{i=1}^n \sigma^2$$

$$= \sigma^2$$

Indeed, it is unbiased.
5.4.15

\[ \mathbb{E}W^2 = \mathbb{E}\left( \frac{1}{n} \sum_{i=1}^{n} W_i \right)^2 \]

\[ = \mathbb{E}\frac{1}{n^2} \left( \sum_{i=1}^{n} W_i^2 + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} W_i W_j \right) \quad \text{Expand} \]

\[ = \frac{1}{n^2} \left( \sum_{i=1}^{n} \mathbb{E}W_i^2 + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \mathbb{E}W_i W_j \right) \quad \text{Linearity} \]

\[ = \frac{1}{n^2} \left[ \sum_{i=1}^{n} (\sigma^2 + \mu^2) + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \mu \mu \right] \quad \text{Independence} \]

\[ = \frac{1}{n^2} \sigma^2 + \mu^2 \quad \text{Plugin moments} \]

Indeed, it is asymptotically unbiased.

5.4.22

Like one midterm problem, you need to compute \( \text{var}(cW_1 + (1 - c)W_2) \) with independence assumption.

\[ f(c) = \text{var}(cW_1 + (1 - c)W_2) = c^2 \sigma_1^2 + (1 - c)^2 \sigma_2^2 = (\sigma_1^2 + \sigma_2^2)c^2 - 2c\sigma_2^2 + \sigma_2^2 \]

Solve critical point

\[ f'(c) = 2(\sigma_1^2 + \sigma_2^2)c - 2\sigma_2^2 = 0 \Rightarrow c = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \]

Since

\[ f''(c) = 2(\sigma_1^2 + \sigma_2^2) > 0 \]

\( c = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \) is the global minimal by convexity.

5.5.2

First check unbiasedness

\[ \mathbb{E}\hat{\lambda} = \mathbb{E}\frac{1}{n} \sum_{i=1}^{n} X_i = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}X_i = \frac{1}{n} \sum_{i=1}^{n} \lambda = \lambda \]
Then compute the variance
\[
\text{var}(\hat{\lambda}) = \text{var}\left(\frac{1}{n} \sum_{i=1}^{n} X_i \right) = \frac{1}{n^2} \sum_{i=1}^{n} \text{var}(X_i) = \frac{1}{n^2} \sum_{i=1}^{n} \lambda = \frac{\lambda}{n}
\]

Second derivative
\[
\frac{\partial^2}{\partial \lambda^2} \ln p_X(k; \lambda) = -\frac{k}{\lambda^2}
\]

The Fisher Information is
\[
I(\lambda) = \mathbb{E} \frac{X}{\lambda^2} = \frac{1}{\lambda}
\]

The Cramér-Rao lower bound:
\[
\frac{1}{nI(\lambda)} = \frac{\lambda}{n} = \text{var}(\hat{\lambda})
\]

Indeed, it is efficient.

5.5.3 Check unbiasedness
\[
\mathbb{E}\hat{\mu} = \mathbb{E}\frac{1}{n} \sum_{i=1}^{n} Y_i = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}Y_i = \frac{1}{n} \sum_{i=1}^{n} \mu = \mu
\]

Compute variance
\[
\text{var}(\hat{\mu}) = \text{var}\left(\frac{1}{n} \sum_{i=1}^{n} Y_i \right) = \frac{1}{n^2} \sum_{i=1}^{n} \text{var}(Y_i) = \frac{1}{n^2} \sum_{i=1}^{n} \sigma^2 = \frac{\sigma^2}{n}
\]

Second derivative
\[
\frac{\partial^2}{\partial \mu^2} \ln f_Y(y; \mu) = -\frac{1}{\sigma^2}
\]

Fisher Information
\[
I(\mu) = \mathbb{E} \frac{1}{\sigma^2} = \frac{1}{\sigma^2}
\]

The Cramér-Rao lower bound:
\[
\frac{1}{nI(\mu)} = \frac{\sigma^2}{n} = \text{var}(\hat{\mu})
\]

Indeed, it is efficient.

5.5.7
Take derivative under integral
\[
\int_{\mathbb{R}} f_{\theta}(y) dy = 1 \iff \int_{\mathbb{R}} \frac{\partial}{\partial \theta} f_{\theta}(y) dy = 0
\]
Notice
\[ \frac{\partial}{\partial \theta} \ln f_\theta(y) = \frac{1}{f_\theta(y)} \frac{\partial}{\partial \theta} f_\theta(y) \]

Then,
\[ \int_\mathbb{R} f_\theta(y) \frac{\partial}{\partial \theta} \ln f_\theta(y) dy = 0 \]

Take derivative under integral again
\[ \int_\mathbb{R} f_\theta(y) \frac{\partial^2}{\partial \theta^2} \ln f_\theta(y) + \frac{\partial}{\partial \theta} f_\theta(y) \frac{\partial}{\partial \theta} \ln f_\theta(y) dy = 0 \]

\[ \Leftrightarrow \int_\mathbb{R} f_\theta(y) \frac{\partial^2}{\partial \theta^2} \ln f_\theta(y) dy = -\int_\mathbb{R} \frac{\partial}{\partial \theta} f_\theta(y) \frac{\partial}{\partial \theta} \ln f_\theta(y) dy \]

Apply the same identity
\[ \Leftrightarrow \int_\mathbb{R} f_\theta(y) \frac{\partial^2}{\partial \theta^2} \ln f_\theta(y) dy = -\int_\mathbb{R} f_\theta(y) \left( \frac{\partial}{\partial \theta} \ln f_\theta(y) \right)^2 dy \]

Which is
\[ \mathbb{E} \frac{\partial^2}{\partial \theta^2} \ln f_\theta(Y) = -\mathbb{E} \left( \frac{\partial}{\partial \theta} \ln f_\theta(Y) \right)^2 \]