Math 181A Homework 6 Solution

Problem: For $X \sim B(n, p)$ derive the 95% for $p$ based on the MLE. Then, compute the observed Fisher information. So $X = \sum_{i=1}^{n} X_i$, each is a Bernoulli($p$).

The Likelihood:

$$L(p) \propto p^X (1 - p)^{n-X}$$

The log-likelihood:

$$l(p) = X \log p + (n - X) \log(1 - p) + C$$

First derivative,

$$l'(p) = \frac{X}{p} - \frac{n - X}{1 - p}$$

We get critical point $\hat{p} = X/n$. Indeed it is the MLE for $l(0+) = l(1-) = -\infty$.

Then, let us derive the Fisher information:

$$I(p) = \mathbb{E} \frac{X_1}{p^2} + 1 - \frac{X_1}{(1-p)^2} = \frac{p}{p^2} + \frac{1 - p}{(1-p)^2} = \frac{1}{p(1-p)}$$

Remember, you must have ”repeated” samples in order to use asymptotic results. In this case, you can do nothing with a single $X$ from binomial. Instead, you need to treat it as $n$ i.i.d. $X_i$ from bernoulli. Of course, there is something to do with the so called ”sufficient statistics”.

The CI we derive from MLE is

$$[\hat{p} - 1.96 \sqrt{\frac{1}{nI(\hat{p})}}, \hat{p} + 1.96 \sqrt{\frac{1}{nI(\hat{p})}}] = [\frac{X}{n} - 1.96 \sqrt{\frac{\frac{X}{n}(1 - \frac{X}{n})}{n}}, \frac{X}{n} + 1.96 \sqrt{\frac{\frac{X}{n}(1 - \frac{X}{n})}{n}}]$$

The observed Fisher information is

$$\mathcal{J}(p) = \frac{1}{n} \sum_{i=1}^{n} - \frac{\partial^2 \log f_{X_i}(x; p)}{\partial p^2} = \frac{X}{np^2} + \frac{n - X}{n(1-p)^2}$$

6.2.2

Distribution: $\mathcal{N}(\mu, 15^2)$. Sample size :22.

Test:

$$H_0 : \mu = 95, H_a : \mu \neq 95, \alpha = 0.06$$
By Theorem 6.2.1, the test statistic is \( Z = \frac{\bar{y} - 95}{15/\sqrt{22}} \).

We reject if \(|Z| > z_{0.03} = 1.88\).

Solve the inequality, \( H_0 \) is rejected if

\[ \bar{y} > 95 + 1.88 \times \frac{15}{\sqrt{22}} = 101 \]

or

\[ \bar{y} < 95 - 1.88 \times \frac{15}{\sqrt{22}} = 89 \]

6.2.5

No. Proof by counter example.

As in Theorem 6.2.1, we use the same statistic \( Z \) to test Normal mean. Suppose we have \( Z = z_{0.75\alpha} \), i.e. \( 0 < z_{\alpha} < Z < z_{\alpha/2} \) In the one-sided level \( \alpha \) test, \( Z > z_{\alpha} \), so \( H_0 \) is rejected. In the two-sided level \( \alpha \) test, \( -z_{\alpha/2} < 0 < Z < z_{\alpha/2} \), so \( H_0 \) is accepted.

6.2.7

(a)

\[ Z = \frac{\bar{y} - 12.6}{0.4/\sqrt{30}} = 2.145 > 1.96 = z_{\alpha/2} \]

\( H_0 \) is rejected in favor of \( H_1 \).

(b)

Distribution of measurement: \( N(\mu, 0.4^2) \). The support of Normal is \( \mathbb{R} \) which does not match with that of a percentage object, \([0, 100] \).

6.2.9

\[ Z = \frac{\bar{y} - 120}{\frac{10}{\sqrt{16}}} = 0.92 = z_{0.179} \]

So, the p-value is \( 0.179 \times 2 = 0.358 \). \( H_0 \) will be rejected if \( \alpha > 0.358 \).

6.2.10 Distribution: \( N(\mu, 12^2) \). Sample size :50.
Test:

\[ H_0 : \mu = 120, H_a : \mu > 120, \alpha = 0.05 \]

By Theorem 6.2.1, the test statistic is

\[ Z = \frac{125.2 - 120}{12/\sqrt{50}} = 3.06 > 1.65 = z_{0.05} \]

We reject \( H_0 \) in favor of \( H_a \). So yes, the pressure makes a difference.