Math 181B Homework 2 Solution

1. Pick an arbitrary $\lambda_1 > \lambda_0$. The likelihood ratio is

$$LR = \frac{\lambda_0^{2n}}{\lambda_1^{2n}} e^{-\sum_{i=1}^{n} x_i (\lambda_0 - \lambda_1)}$$

Since $\lambda_0 - \lambda_1 < 0$, $LR$ is an increasing function of (its sufficient statistic) $\sum_{i=1}^{n} x_i$.

By Neyman-Pearson lemma, the Most Powerful level $\alpha$ test for $H_0^*: \lambda = \lambda_0$ versus $H_1^*: \lambda = \lambda_1$ is given by rule

$$LR < C \iff \sum_{i=1}^{n} x_i < C^*$$

with size $\alpha$.

By the property of Gamma distribution

$$\sum_{i=1}^{n} x_i \sim Gamma(2n, \lambda_0)$$

so $C^*$ is the $1 - \alpha$ quantile of $Gamma(2n, \lambda_0)$.

Since the test is all the same for any $\lambda_1$, i.e. does not depend on our choice of $\lambda_1$, we have this single test Most Powerful at any $\lambda_1 > \lambda_0$. That is the UMP.

2. (a) Warning: you cannot reduce $H_0$ at the beginning!

Pick an arbitrary $\mu_1 > \mu_0$. Then, we have this ”Many versus Simple” test

$$H_0: \mu \leq \mu_0 \text{ vs. } H_1: \mu = \mu_1$$

By a result from the lecture, you may now reduce the $H_0$ to a simple case. That is: the MP test for ”Many versus Simple” above is the same as the MP test for ”Simple vs. Simple”

$$H_0: \mu = \mu_0 \text{ vs. } H_1: \mu = \mu_1$$

The likelihood ratio is

$$LR = e^{-\frac{1}{2\sigma_0^2} [2(\mu_1 - \mu_0) \sum_{i=1}^{n} x_i + \mu_0^2 - \mu_1^2]}$$

It is a decreasing function of $\sum_{i=1}^{n} x_i$, so

$$LR < C \iff \sum_{i=1}^{n} x_i > C^*$$

to achieve a size of $\alpha$, $C^* = n\mu_0 + \sqrt{n}\sigma_0\Phi(1 - \alpha)$.

By Neyman-Pearson lemma, that is the MP test.

Again, the test is the same for all $\mu_1 > \mu_0$, so it is the UMP.
(b) Pick \( \mu_1 = \mu_0 + 1 \) and \( \mu_2 = \mu_0 - 1 \) from the alternative.

For test

\[ H_0 : \mu = \mu_0 \text{ vs. } H_1 : \mu = \mu_1 \]

The MP test is given by \( \sum_{i=1}^{n} x_i > C^* \).

In the other hand,

\[ H_0 : \mu = \mu_0 \text{ vs. } H_1 : \mu = \mu_2 \]

has MP test \( \sum_{i=1}^{n} x_i < 2n\mu_0 - C^* \).

There is an ”almost sure” uniqueness of MP test. Since the two rejection regions are disjoint and each of probability \( \alpha \), there is no set of probability \( \alpha \) under null that can equal to them almost surely. Thus, there is no UMP test.

Intuitively, think of a bargain between acceptance and rejection. For a certain point in the probability space, likelihood under null is its ”price” of rejection while the likelihood under alternative is its ”gain” of rejection. Then, if we have fix budget, we will always go for points with high ”return rate”.

3. (a) Under the given conditions, the likelihood is

\[
L(\mu_X, \mu_Y, \sigma^2) = \exp\left\{ -\frac{\sum_{i=1}^{n}(x_i - \mu_X)^2 - 2\rho(x_i - \mu_X)(y_i - \mu_Y) + (y_i - \mu_Y)^2}{2\sigma^2(1 - \rho^2)} \right\} \frac{1}{(2\pi\sigma^2)^\frac{n}{2}}
\]

Under \( H_0 \), log-likelihood

\[
l(\mu, \sigma^2) \propto -n \log \sigma^2 - \frac{1}{2\sigma^2(1 - \rho^2)} \sum_{i=1}^{n} (x_i - \mu)^2 - 2\rho(x_i - \mu)(y_i - \mu) + (y_i - \mu)^2
\]

Take partial w.r.t \( \mu \),

\[
\frac{\partial l}{\partial \mu} = \frac{1}{2\sigma^2(1 - \rho^2)} \sum_{i=1}^{n} 2(x_i - \mu) - 2\rho(x_i - \mu) - 2\rho(y_i - \mu) + 2(y_i - \mu) = 0
\]

\[
\Rightarrow \hat{\mu} = \frac{\bar{X} + \bar{Y}}{2}
\]

Then, replace \( \mu \) by \( \hat{\mu} \) and take derivative w.r.t. \( \sigma^2 \).

\[
\frac{\partial l}{\partial \sigma^2} = -\frac{n}{\sigma^2} + \frac{1}{2\sigma^4(1 - \rho^2)} \sum_{i=1}^{n} (x_i - \hat{\mu})^2 - 2\rho(x_i - \hat{\mu})(y_i - \hat{\mu}) + (y_i - \hat{\mu})^2 = 0
\]

\[
\hat{\sigma}^2 = \frac{1}{2n(1 - \rho^2)} \sum_{i=1}^{n} (x_i - \hat{\mu})^2 - 2\rho(x_i - \hat{\mu})(y_i - \hat{\mu}) + (y_i - \hat{\mu})^2
\]
Under $H_1$, partial derivative w.r.t. $\mu_X, \mu_Y$:
\[
\begin{align*}
\frac{\partial l}{\partial \mu_X} &= \frac{1}{2\sigma_1^2(1-\rho^2)} \sum_{i=1}^{n} 2(x_i - \mu_X) - 2\rho(y_i - \mu_Y) = 0 \\
\frac{\partial l}{\partial \mu_Y} &= \frac{1}{2\sigma_1^2(1-\rho^2)} \sum_{i=1}^{n} -2\rho(x_i - \mu_X) + 2(y_i - \mu_Y) = 0
\end{align*}
\]
\[
\Rightarrow \begin{cases} \\
\hat{\mu}_X = \bar{X} \\
\hat{\mu}_Y = \bar{Y}
\end{cases}
\]
Similarly,
\[
\hat{\sigma}_1^2 = \frac{1}{2n(1-\rho^2)} \sum_{i=1}^{n} (x_i - \bar{X})^2 - 2\rho(x_i - \bar{X})(y_i - \bar{Y}) + (y_i - \bar{Y})^2
\]
The likelihood ratio is
\[
\Lambda = \left( \frac{\hat{\sigma}_1^2}{\hat{\sigma}_0^2} \right)^n
\]
(b) Obviously, $\Lambda$ is an increasing function of $T = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_0^2}$.

The following trick comes from the idea of "projection". You will learn it in detail as ANOVA.
\[
T - 1 = \frac{\sum_{i=1}^{n}(x_i - \hat{\mu})^2 - 2\rho(x_i - \hat{\mu})(y_i - \hat{\mu}) + (y_i - \hat{\mu})^2}{\sum_{i=1}^{n}(x_i - \bar{X})^2 - 2\rho(x_i - \bar{X})(y_i - \bar{Y}) + (y_i - \bar{Y})^2} - 1
\]
\[
= \frac{n(1+\rho)(\bar{X} - \bar{Y})^2/2}{2n(1-\rho^2)^2/2}
\]
\[
\propto \left( \frac{\bar{Z}}{\hat{se}(Z)} \right)^2
\]
Where $Z_i = X_i - Y_i \overset{H_0}{\sim} \mathcal{N}(0, 2(1-\rho)\sigma^2)$. By the hint given by Dr Bradić,
\[
\frac{Z}{\hat{se}(Z)} \sim t_{2n-2}
\]
(c) No. We don’t assume $\rho$ as known or $\sigma_X^2 = \sigma_Y^2$ for paired t-test. In first step, get $Z_i = X_i - Y_i \sim \mathcal{N}(\mu_Z, \sigma_Z^2)$. Then, forget about $X_i, Y_i, etc..$ Test if $\mu_Z = 0$ directly.

4. See code.