Math 181B Worksheet Week 3

Homework Hint for Problem 3

- For normal with unknown mean and variance, you should always settle $\hat{\mu}$ first. Then, plug that in and solve $\hat{\sigma}^2$.
- The likelihood ratio can be simplified as $\frac{\hat{\sigma}^2_0}{\hat{\sigma}^2}$.
- Find the distribution of $Z_i = X_i - Y_i$ and notice the studentized statistic
  \[
  \frac{\hat{\sigma}^2_0}{\sigma^2} - 1 = \frac{\bar{Z}}{\hat{se}(\bar{Z})}
  \]
  where $\hat{se}$ stands for replacing true $\sigma^2$ by $\hat{\sigma}^2$ in the standard deviation. You may claim without proof that it is $t_{n-1}$ distributed. (Proof will be available in solution Appendix)

One Sided test for Exponential Family

- Exponential family:
  \[
  f_\theta(x) = a(x)e^{gT(x_i) - \psi(\theta)}
  \]
- Examples: Binomial, Poisson, Exponential, Normal, Gamma, Beta, etc.
- One sided UMP test: likelihood ratio is always a monotonic function of sufficient statistics.
- A list of sufficient statistics is presented on the back side. (You don’t have to know the meaning of sufficiency since we skipped that section in 181A.)

Practice: Find the UMP tests

1. $X_1, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$
   \[
   H_0 : \lambda = 1 \text{ versus } H_1 : \lambda \geq 1
   \]

2. $X \sim \text{Binomial}(n,p)$
   \[
   H_0 : p = 0.5 \text{ versus } H_1 : p > 0.5
   \]

3. $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, 1)$
   (a) $H_0 : \mu = 0 \text{ versus } H_1 : \mu = 1$
   (b) $H_0 : \mu = 0 \text{ versus } H_1 : \mu = -1$
   (c) Use the two tests above to show the nonexistence of UMP test for
   \[
   H_0 : \mu = 0 \text{ versus } H_1 : \mu \neq 0
   \]
List of Sufficient Statistics

Suppose we have i.i.d. samples $X_1, \ldots, X_n$ from the following distributions. The sufficient statistic is denoted as $T$.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Sufficient Statistic $T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform$(0, \theta)$</td>
<td>$X_{(n)} = \max{X_1, \ldots, X_n}$</td>
</tr>
<tr>
<td>Normal$(\mu, 1)$</td>
<td>$\bar{X}$</td>
</tr>
<tr>
<td>Normal$(0, \sigma^2)$</td>
<td>$\sum_{i=1}^{n} X_i^2$</td>
</tr>
<tr>
<td>Normal$(\mu, \sigma^2)$</td>
<td>$(\bar{X}, \sum_{i=1}^{n} (X_i - \bar{X})^2)$</td>
</tr>
<tr>
<td>Poisson$(\lambda)$</td>
<td>$\sum_{i=1}^{n} X_i$</td>
</tr>
<tr>
<td>Binomial$(p)$</td>
<td>$\bar{X}$</td>
</tr>
<tr>
<td>Exp$(\theta)$</td>
<td>$\bar{X}$</td>
</tr>
<tr>
<td>Gamma$(\alpha, \beta)$</td>
<td>$(\prod_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i)$</td>
</tr>
</tbody>
</table>