Midterm Review

1. Let \((X_1, \ldots, X_n)\) and \((Y_1, \ldots, Y_m)\) be two Exponential Distribution with parameter \(\lambda_1\) and \(\lambda_2\) respectively. \(f_{X}(x) = \lambda \exp\{-\lambda x\}\)

Design LRT for the following testing problem

\[ H_0 : \frac{\lambda_1}{\lambda_2} = c \text{ versus } H_1 : \frac{\lambda_1}{\lambda_2} > c \]

and compute the approximate critical region of 5% significance level.

**Solution:**

The likelihood function looks like:

\[ L(\lambda_1, \lambda_2) = \lambda_1^n \lambda_2^m e^{-\lambda_1 \sum_{i=1}^n x_i - \lambda_2 \sum_{i=1}^m y_i} \]

The log-likelihood:

\[ l(\lambda_1, \lambda_2) = n \log \lambda_1 + m \log \lambda_2 - \lambda_1 \sum_{i=1}^n x_i - \lambda_2 \sum_{i=1}^m y_i \]

Under \(H_0\), \(\frac{\lambda_1}{\lambda_2} = c \Rightarrow \lambda_1 = c \lambda_2\). The log-likelihood:

\[ l(\lambda_2) = (n + m) \log \lambda_2 + n \log c - \lambda_2 (c \sum_{i=1}^n x_i + \sum_{i=1}^m y_i) \]

1st derivative:

\[ l'(\lambda_2) = \frac{n + m}{\lambda_2} - (c \sum_{i=1}^n x_i + \sum_{i=1}^m y_i) \]

2nd derivative:

\[ l''(\lambda_2) = -\frac{n + m}{\lambda_2^2} \]

Critical point \(\hat{\lambda}_2 = \frac{n + m}{c \sum_{i=1}^n x_i + \sum_{i=1}^m y_i}\) from \(l' = 0\). Since \(l''(\lambda_2) < 0\), the concave function must have its maximal at the unique critical point.

Under \(H_1\), things are more complicated.

First, let us try to maximize the log-likelihood without any restriction. The gradient:

\[ \nabla l = \begin{cases} 
\frac{\partial l}{\partial \lambda_1} = \frac{n}{\lambda_1} - \sum_{i=1}^n x_i \\
\frac{\partial l}{\partial \lambda_2} = \frac{m}{\lambda_2} - \sum_{i=1}^m y_i 
\end{cases} \]
Then its Hessian is

\[ \nabla^2 l = \begin{pmatrix} -\frac{n}{\lambda_1^2} & 0 \\ 0 & -\frac{m}{\lambda_2^2} \end{pmatrix} \]

Again, since \( \nabla^2 l \) is negative definite, \( l \) is concave. Its maximal is reached by its critical point \( \hat{\lambda}_1 = \frac{n}{\sum_{i=1}^{n} x_i} \), \( \hat{\lambda}_2 = \frac{m}{\sum_{i=1}^{m} y_i} \).

Case 1: if \( \hat{\lambda}_1/\hat{\lambda}_2 \geq c \), then we shall be happy to have the global maximal \( (\hat{\lambda}_1, \hat{\lambda}_2) \) as the maximal under \( H_1 \).

Case 2: if \( \hat{\lambda}_1/\hat{\lambda}_2 < c \), let us look at the gradient \( \nabla l \) one more time.

For any \( \lambda_1 > c\lambda_2 \), either \( \lambda_1 > \hat{\lambda}_1 \) or \( \lambda_2 < \hat{\lambda}_2 \). (otherwise, manipulate the inequalities to get a contradiction.)

Therefore, either

\[ \frac{\partial l}{\partial \lambda_1} = \frac{n}{\lambda_1} - \sum_{i=1}^{n} x_i < \frac{n}{\lambda_1} - \sum_{i=1}^{n} x_i = 0 \Rightarrow l(c\lambda_2, \lambda_2) > l(\lambda_1, \lambda_2) \]

or

\[ \frac{\partial l}{\partial \lambda_2} = \frac{m}{\lambda_2} - \sum_{i=1}^{m} y_i > \frac{m}{\lambda_2} - \sum_{i=1}^{m} y_i = 0 \Rightarrow l(\lambda_1, \lambda_1/c) > l(\lambda_1, \lambda_2) \]

Thus, the maximal of \( l(\lambda_1, \lambda_2) \) over \( \{ (\lambda_1, \lambda_2) : \lambda_1 \geq c\lambda_2 \} \) must be achieved at its boundary \( \{ (\lambda_1, \lambda_2) : \lambda_1 = c\lambda_2 \} \), which is already studied for \( H_0 \).

The likelihood ratio is therefore:

\[ \Lambda = \begin{cases} \left( \frac{c \sum_{i=1}^{n} x_i + \sum_{i=1}^{m} y_i}{\sum_{i=1}^{n} x_i} \right)^{m+n} \frac{n \sum_{i=1}^{m} y_i}{m \sum_{i=1}^{n} x_i} > c & \\
1 & \frac{n \sum_{i=1}^{m} y_i}{m \sum_{i=1}^{n} x_i} \leq c \end{cases} \]

Write \( \sum_{i=1}^{n} x_i = u \),

\[ \Lambda = \begin{cases} \frac{(n + m)^{m+n}}{n^m m^n} \frac{u^n}{(cu + 1)^{n+m}} & u < \frac{n}{cm} \\
1 & u \geq \frac{n}{cm} \end{cases} \]

\( \Lambda \) is a monotone increasing continuous function of \( u \).

\( H_0 \) is rejected when \( u < C^* \) for some \( C^* \).

Finally, we have to figure out the distribution of \( u \).

Under \( H_0 \), \( \sum_{i=1}^{n} x_i \sim \text{Gamma}(n, 1/(c\lambda_2)) \) and \( \sum_{i=1}^{m} y_i \sim \text{Gamma}(m, 1/\lambda_2) \). Note that both of them depend on the choice of \( \lambda_2 \).
Then, we have to rewrite \( u = \frac{\lambda_2 \sum_{i=1}^{n} x_i}{\lambda_2 \sum_{i=1}^{m} y_i} \) so that \( \lambda_2 \sum_{i=1}^{n} x_i \sim \text{Gamma}(n, 1/c) \) and \( \lambda_2 \sum_{i=1}^{m} y_i \sim \text{Gamma}(m, 1) \).

2. Story omitted. \((16, 398, 3, 225) \sim \text{Multinomial}(642, pq, (1 − p)q, p(1 − q), (1 − p)(1 − q))\)

(a) Compute the MLE for \( p \) and \( q \).

(b) Do Pearson’s goodness of test using \( \hat{p} \) and \( \hat{q} \).

Solution:

(a) Compute the MLE for \( p \) and \( q \).

Likelihood:

\[
L(p, q) = \left( \frac{642}{16 398 3 225} \right)^p (1 - p)^{623} q^{414} (1 - q)^{228}
\]

Similar to the binomial MLE, \( \hat{p} = 19/642 = 0.03 \) and \( \hat{q} = 0.64 \)

(b) Do Pearson’s goodness of test using \( \hat{p} \) and \( \hat{q} \). The estimated expected counts are

\[
E_1 = 642 \hat{p} \hat{q} = 12.25, \quad E_2 = 642(1 - \hat{p}) \hat{q} = 401.75,
\]

\[
E_3 = 642 \hat{p}(1 - \hat{q}) = 6.75, \quad E_4 = 642(1 - \hat{p})(1 - \hat{q}) = 221.25,
\]

\[
X^2 = 2.4979 < 3.8, \text{ so } H_0 \text{ accepted.}
\]