Concepts

- **Least Square Solution**
  For linear model
  \[ Y = Xb + \epsilon \]
  where \( Y \in \mathbb{R}^n \), \( b \in \mathbb{R}^p \), \( X \in \mathbb{R}^{n \times p} \), \( \epsilon \sim \mathcal{N}(0, \sigma^2 I_n) \)
  Suppose \( \text{rank}(X) = p \), the least square estimator is given by
  \[ \hat{b}_{LS} = (X'X)^{-1}X'Y \]

- **Mean and Variance of a Vector**
  Let \( v \) be a \( n \)-dim random vector, \( A \) is a \( p \times n \) matrix. Then
  \[ EAv = AEv, \text{Var}Av = A\text{Var}[v]A^T \]

Problems

1. Suppose \( \hat{\beta} = \phi'Y \) is an unbiased linear estimator for \( \beta \). Compute its Mean-Square Error.

2. (Shrinkage) Now consider a simple linear model \( Y_i \sim \beta_0 + \beta_1 X_i \), where \( \sum_{i=0}^n X_i = 0 \) and \( \sum_{i=0}^n X_i^2 = 1 \).
   (a) Derive the LS estimator \( \hat{\beta}_{1LS} \).

   (b) Let \( \hat{\beta}_1^\lambda = \hat{\beta}_{1LS} / (1 + \lambda) \), compute its MSE.

   (c) Show that if \( \beta_1 \neq 0 \), \( \exists \lambda > 0 \), s.t. \( MSE(\hat{\beta}_1^\lambda) < MSE(\hat{\beta}_{1LS}) \)