1. Let $\mathcal{F} = \{f \in H(D) | f(\frac{1}{2}) = 0 \text{ and } \sup_{z \in D} |f(z)| \leq 1\}$. Compute $\sup_{f \in \mathcal{F}} |f(\frac{1}{4})|$.

*Solution.* Note that if $g(z) = f(\phi_{-1/2}(z))$, then $f(\frac{1}{2}) = 0$ if and only if $g(0) = 0$. Therefore, since $\phi_{-1/2}$ is an analytic bijection of the disc with inverse $\phi_{1/2}$,

$$\sup_{f \in \mathcal{F}} |f(\frac{1}{4})| = \sup_{g \in \mathcal{G}} |g(\phi_{1/2}(\frac{1}{4}))|,$$

where $\mathcal{G} = \{g \in H(D) | g(0) = 0 \text{ and } \sup_{z \in D} |g(z)| \leq 1\}$. Since, $\phi_{1/2}(\frac{1}{4}) = -\frac{2}{7}$ and Schwarz’s Lemma implies that

$$\sup_{g \in \mathcal{G}} |g(-\frac{2}{7})| = \frac{2}{7},$$

it follows that $\sup_{f \in \mathcal{F}} |f(\frac{1}{4})| = \frac{2}{7}$.

2. Let $G$ be an open set in $\mathbb{C}$ and let $\mathcal{F} \subseteq H(G)$. Prove that if $\mathcal{F}$ is locally bounded, then $\mathcal{F}$ is locally Lipschitz.

*Solution.* Fix $a \in G$. Since $\mathcal{F}$ is assumed locally bounded, there exists $r > 0$ and a constant $M$ such that $B(0; r)^- \subseteq G$ and

$$\forall f \in \mathcal{F} \forall \lambda \in B(a; r^-) \quad |f(\lambda)| \leq M.$$  

We claim that $\mathcal{F}$ is uniformly Lipschitz on $B(a; r/2)$. To see this fix $z, w \in B(a; r/2)$ and let $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$. Then by the Cauchy Integral
Formula, for each \( f \in \mathcal{F} \),

\[
|f(z) - f(w)| = \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(\lambda)}{\lambda - z} d\lambda - \frac{1}{2\pi i} \int_{\gamma} \frac{f(\lambda)}{\lambda - w} d\lambda \right|
\]

\[
= \frac{1}{2\pi} \int_{\gamma} \frac{(z - w)f(\lambda)}{(\lambda - z)(\lambda - w)} d\lambda
\]

\[
\leq \frac{|z - w|}{2\pi} \max_{\lambda \in [\gamma]} \left| \frac{f(\lambda)}{(\lambda - z)(\lambda - w)} \right| \|\gamma\|
\]

\[
= \frac{|z - w|}{2\pi} \frac{M}{\frac{r}{2}} 2\pi r
\]

\[
= \frac{4M}{r} |z - w|.
\]

This proves that \( \mathcal{F} \) is uniformly Lipshitz on \( B(a; r/2)^- \) as was claimed.

3. Show that \( \mathcal{F} = \{ f \in H(\mathbb{D}) | f(0) = 1 \text{ and } \forall z \in \mathbb{D} \ \text{Re} f(z) > 0 \} \) is a normal family.

**Solution.** By Exercise #3 pg. 133, if \( f \in \mathcal{F} \) and \( z \in \mathbb{D} \), then

\[
|f(z)| \leq \frac{1 + |z|}{1 - |z|}.
\]

Hence, if \( K \) is a compact subset of \( \mathbb{D} \),

\[
\sup_{f \in \mathcal{F}} \max_{z \in K} |f(z)| \leq \max_{z \in K} \frac{1 + |z|}{1 - |z|}.
\]

This proves that \( \mathcal{F} \) is locally bounded. It follows from Montel’s Theorem that \( \mathcal{F} \) is normal.

**Alternate Solution.** Let \( T(z) = \frac{z - 1}{z + 1} \) so that \( T \) is a conformal map from \( \{ z \in \mathbb{C} | \text{Re} z > 0 \} \) onto \( \mathbb{D} \) satisfying \( T(1) = 0 \). Fix a sequence \( \{ f_n \} \) in \( \mathcal{F} \) and for each \( n \) set \( g_n = T \circ f_n \). Since for each \( n \), \( \sup_{z \in \mathbb{D}} |g_n(z)| \leq 1 \), Montel’s Theorem implies that there exists \( g \in H(\mathbb{D}) \) and a subsequence \( |g_{n_k}| \) such that \( g_{n_k} \to g \) in \( H(\mathbb{D}) \).

Now, since \( f_n(0) = 1 \) for each \( n \), \( g_n(0) = 0 \) for each \( n \). Hence, \( g(0) = 0 \). Also, since for each \( n \) \( \sup_{z \in \mathbb{D}} |g_n(z)| \leq 1 \), \( \sup_{z \in \mathbb{D}} |g(z)| \leq 1 \). Therefore, by the Maximum Principle, for each \( z \in \mathbb{D} \) we have that \( |g(z)| < 1 \), i.e.
\( f = T^{-1} \circ g \in \mathcal{F} \). Since, \( T \) transforms compact subsets of \( \{ z \in \mathbb{C} \mid \text{Re} \ z > 0 \} \) to compact subsets of \( \mathbb{D} \) and \( g_{n_k} \) converges uniformly to \( g \) on compact subsets of \( \mathbb{D} \), \( f_{n_k} \) converges uniformly to \( f \) on compact subsets of \( \{ z \in \mathbb{C} \mid \text{Re} \ z > 0 \} \).

Summarizing, we have shown that if \( \{ f_n \} \) is a sequence in \( \mathcal{F} \), then there exists \( f \in H(\{ z \in \mathbb{C} \mid \text{Re} \ z > 0 \}) \) and a subsequence \( \{ f_{n_k} \} \) such that \( f_{n_k} \rightarrow f \) in \( H(\{ z \in \mathbb{C} \mid \text{Re} \ z > 0 \}) \), i.e., \( \mathcal{F} \) is normal.

4. Let \( G \) be a simply connected region in the plane and assume that \( f \) is a conformal map from \( G \) to \( \mathbb{D} \) (i.e., \( f : G \rightarrow \mathbb{D} \) is an analytic bijection). Prove that if \( g \) is any other conformal map from \( G \) to \( \mathbb{D} \) then there exist \( c, \alpha \in \mathbb{C} \) with \( |c| = 1 \) and \( \alpha \in \mathbb{D} \) such that \( g(z) = c \phi_{\alpha}(f(z)) \).

\textit{Solution.} A basic fact in the text (Theorem 2.5 pg. 132) was that \( h : \mathbb{D} \rightarrow \mathbb{D} \) is a conformal map from the disc to the disc if and only if there exist \( c, \alpha \in \mathbb{C} \) with \( |c| = 1 \) and \( \alpha \in \mathbb{D} \) such that \( h(w) = c \phi_{\alpha}(w) \) for all \( w \in \mathbb{D} \). Therefore, if \( f \) and \( g \) are two conformal maps from \( G \) to \( \mathbb{D} \), then, as \( g \circ f^{-1} \) is a conformal map from \( \mathbb{D} \) to \( \mathbb{D} \), there exist \( c, \alpha \in \mathbb{C} \) with \( |c| = 1 \) and \( \alpha \in \mathbb{D} \) such that \( g \circ f^{-1}(w) = c \phi_{\alpha}(w) \) for all \( w \in \mathbb{D} \). The desired result follows by letting \( w = f(z) \).

5. Prove that for each \( \epsilon > 0 \), \( \frac{1}{z+i} + \sin z \) has infinitely many zeros in the region \( \{ z = x + iy \mid x > 0, |y| < \epsilon \} \).

\textit{Solution.} Let \( \epsilon > 0 \) where without loss of generality we assume that
\( \sinh \epsilon \leq 1. \) Observe that for \( z = x + iy, \)
\[
\sin z = \frac{1}{2i} (e^{iz} - e^{-iz}) \\
= \frac{1}{2i} (e^{ix-y} - e^{-ix+y}) \\
= \frac{1}{2i} (\cos x + i \sin x) e^{-y} - (\cos x - i \sin x) e^{y} \\
= \frac{1}{2i} \left( \cos x (e^{-y} - e^{y}) + i \sin x (e^{-y} + e^{y}) \right) \\
= \sin x \cosh y + i \cos x \sinh y.
\]

Hence,
\[
|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y. \tag{1}
\]

For \( m, n \) positive integers with \( m < n \) define a closed contour \( C_{m,n} \) by
\[
C_{m,n} = [2m\pi + \frac{\pi}{2} + \epsilon i, 2m\pi + \frac{\pi}{2} - \epsilon i] \\
+ [2m\pi + \frac{\pi}{2} - \epsilon i, 2n\pi + \frac{\pi}{2} - \epsilon i] \\
+ [2n\pi + \frac{\pi}{2} - \epsilon i, 2n\pi + \frac{\pi}{2} + \epsilon i] \\
+ [2n\pi + \frac{\pi}{2} + \epsilon i, 2m\pi + \frac{\pi}{2} + \epsilon i]
\]

Noting that \( \cosh^2 y \geq 1 \) for all \( y \) and \( \sin x = 1 \) on the vertical sides of \( C_{m,n} \), we see using (1) that
\[
|\sin z| \geq 1 \geq \sinh \epsilon
\]
on the vertical sides of \( C_{m,n} \). On the other hand on the horizontal sides of \( C_{m,n} \), (1) implies that
\[
|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\
\geq \sin^2 x + \cos^2 x \sinh^2 \epsilon \\
\geq \sin^2 x \sinh^2 \epsilon + \cos^2 x \sinh^2 \epsilon \\
= \sinh^2 \epsilon.
\]

Therefore, for all choices of positive integers \( m < n \),
\[
|\sin z| \geq \sinh \epsilon \quad \text{for all } z \in C_{m,n}. \tag{2}
\]

Now, since
\[
\max_{z \in C_{m,n}} \left| \frac{1}{z + i} \right| = \frac{1}{m + 1} \to 0 \text{ as } m \to \infty,
\]
it follows that we may choose $m$ such that

$$
\max_{z \in C_{m,n}} \left| \frac{1}{z+i} \right| < \sinh \epsilon \quad \text{for all } n > m.
$$

For this choice of $m$, (2) implies that if $n > m$

$$
\left| \frac{1}{z+i} + \sin z \right| - \sin z = \left| \frac{1}{z+1} \right| < \sinh \epsilon \leq \left| \sin z \right|
$$

for all $z \in C_{m,n}$. Hence, by Rouche’s Theorem, $\frac{1}{z+i} + \sin z$ and $\sin z$ have the same number of $0$'s inside $C_{m,n}$. Since $\sin z$ has $n - m$ 0's inside $C_{m,n}$ and $n > m$ is arbitrary it follows that $\frac{1}{z+i} + \sin z$ has infinitely many zeros in the region $\{ z = x + iy \mid x > 0, |y| < \epsilon \}$. 