Math 220B Preliminary Exam

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Instructions: Solve 4 problems.

1. \( \mathbb{D} \) denotes the unit disc centered at the origin.

2. If \( G \) is an open set in the plane, \( H(G) \) denotes the collection of analytic functions on \( G \).

3. A set \( \mathcal{F} \subseteq H(G) \) is said to be locally Lipschitz if for each \( a \in G \) there exist \( r > 0 \) and a constant \( M \) such that \( B(a;r) \subseteq G \) and \( |f(z) - f(w)| \leq M|z - w| \) for all \( z, w \in B(a;r) \).

4. \( \phi_\alpha \) denotes the Moebius function defined by \( \phi_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha}z} \).

Problems:

1. Let \( \mathcal{F} = \{ f \in H(\mathbb{D}) \mid f\left(\frac{1}{2}\right) = 0 \text{ and } \sup_{z \in \mathbb{D}} |f(z)| \leq 1 \} \). Compute \( \sup_{f \in \mathcal{F}} |f\left(\frac{1}{2}\right)| \).

2. Let \( G \) be an open set in \( \mathbb{C} \) and let \( \mathcal{F} \subseteq H(G) \). Prove that if \( \mathcal{F} \) is locally bounded, then \( \mathcal{F} \) is locally Lipschitz.

3. Show that \( \mathcal{F} = \{ f \in H(\mathbb{D}) \mid f(0) = 1 \text{ and } \forall z \in \mathbb{D} \text{ Re } f(z) > 0 \} \) is a normal family.

4. Let \( G \) be a simply connected region in the plane and assume that \( f \) is a conformal map from \( G \) to \( \mathbb{D} \) (i.e., \( f : G \to \mathbb{D} \) is an analytic bijection). Prove that if \( g \) is any other conformal map from \( G \) to \( \mathbb{D} \) then there exist \( c, \alpha \in \mathbb{C} \) with \( |c| = 1 \) and \( \alpha \in \mathbb{D} \) such that \( g(z) = c\phi_\alpha(f(z)) \).

5. Prove that for each \( \epsilon > 0 \), \( \frac{1}{z+i} + \sin z \) has infinitely many zeros in the region \( \{ z = x + iy \mid x > 0, |y| < \epsilon \} \).