Math 220 B Practice Final Exam Solutions

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General Comments: As these solutions are partly pedagogic in nature, they are a bit less terse than what you should aim for in your solutions. Roughly, depending on the problem, an ideal solution might contain anywhere from 10% to 40% fewer details than I include here. Of course, when in doubt, it is always wise to err on the side of including to many, rather than too few, details in your solutions.

1. Let $G$ be a connected open set in $\mathbb{C}$, $f$ a nonconstant analytic function on $G$, and $a \in G$. For $0 \leq r < \text{dist}(a, \mathbb{C} \setminus G)$, define $M(r)$ by

$$M(r) = \max_{0 \leq \theta < 2\pi} |f(a + re^{i\theta})|.$$ 

Prove that $M$ is a strictly increasing function. Is this result still true if $G$ is open but not connected?

Solution: For $0 \leq r < \text{dist}(a, \mathbb{C} \setminus G)$, let $D_r = \{z | |z| \leq r\}$ and $C_r = \{z | |z| = r\}$. As $G$ is connected, and $f$ is not constant on $G$, $f$ is a nonconstant analytic function defined on a neighborhood of $D_r$. Hence, by the Maximum Principle, if $0 \leq s < t < \text{dist}(a, \mathbb{C} \setminus G)$, then

$$M(s) = \max_{z \in C_s} |f(z)| < \max_{z \in D_t} |f(z)| = \max_{z \in C_t} |f(z)| = M(t).$$

The result is not true if $G$ is not connected. Let $G = G_1 \cup G_2$ where $G_1$ and $G_2$ are disjoint nonempty open sets. Define $f$ on $G$ by setting $f(z) = 0$ for $z \in G_1$ and setting $f(z) = 1$ for $z \in G_2$. Then $f$ is a nonconstant analytic function on $G$ but for any choice of $a \in G$, $M$ is constant, and hence, not
strictly increasing.

2. Let $F$ consist of the analytic functions on $D$ satisfying $f(1/3) = 0$ and $|f(z)| < 1$ for all $z \in D$. Show that $F$ is a compact subset of $H(D)$ and give a brief explanation how this implies that the two suprema,

$$M_1 = \sup_{f \in F} |f(2/3)| \quad \text{and} \quad M_2 = \sup_{f \in F} |f'(1/3)|,$$

are attained.

Solution: As, $|f(z)| < 1$ for all $z \in D$ whenever $f \in F$, Montel’s Theorem implies that $F$ is normal in $H(G)$. We claim that $F$ is closed in $H(G)$. To see this, let $f_n \in F$ with $f_n \rightarrow f$ in $H(G)$. Note that $f(1/3) = \lim_{n \rightarrow \infty} f_n(1/3) = 0$. Also, if $z \in D$, then $|f(z)| = \lim_{n \rightarrow \infty} |f_n(z)| \leq 1$. If for some $z_0 \in D$, $|f(z_0)| = 1$, then by the Maximum Principle, $f$ is constant so that $f \equiv f(1/3) = 0$ and $f \in F$. Otherwise, $|f(z)| < 1$ for all $z \in D$ and again, $f \in F$. This proves that $F$ is closed.

As $F$ is both normal and closed in $H(G)$, it follows that $F$ is a compact subset of $H(G)$. Furthermore, the two functions, $F_1, F_2 : F \rightarrow \mathbb{R}$ defined by $F_1(f) = |f(2/3)|$ and $F_1(f) = |f'(1/3)|$ are continuous. As the extrema of continuous functions on compact sets are attained (Extreme Value Theorem), it follows that the extrema, $M_1$ and $M_2$ are attained.

Comment: Note that the idea in this problem is identical to the idea that used in the proof of the Riemann Mapping Theorem, where, at the key moment, the key fact was that the set $F$ defined on the middle of page 161 of the text had the property that $F \cup \{0\}$ was a nonempty compact set.

3. Compute the $M_1$ and $M_2$ that are defined in the previous problem.

Solution: We move the point $1/3$ to 0 using a Moebius transformation of $D$ and then invoke Schwarz’s Lemma. Recall that for $a \in D$, $\phi_a$ is defined by

$$\phi_a(z) = \frac{z - a}{1 - \overline{a}z}$$

is an analytic bijection from $D$ to $D$ with the property that $\phi_a(a) = 0$. Hence,
if we define $\mathcal{G}$ to consist of the analytic functions $g$ on $\mathbb{D}$ satisfying $g(0) = 0$ and $|g(z)| < 1$ for all $z \in \mathbb{D}$, then

$$f \in \mathcal{F} \iff \exists g \in \mathcal{G} \ f = g \circ \phi_{\frac{1}{3}}.$$ 

Consequently, noting that Schwarz’s Lemma implies that $\sup_{g \in \mathcal{G}} |g(\frac{2}{3})| = \frac{3}{7}$, we see that

$$M_1 = \sup_{f \in \mathcal{F}} |f(\frac{2}{3})|$$

$$= \sup_{g \in \mathcal{G}} |(g \circ \phi_{\frac{1}{3}})(\frac{2}{3})|$$

$$= \sup_{g \in \mathcal{G}} |g(\phi_{\frac{1}{3}}(\frac{2}{3}))|$$

$$= \sup_{g \in \mathcal{G}} |g(\frac{3}{7})|$$

$$= \frac{3}{7}.$$ 

Likewise, as Schwarz’s Lemma implies that $\sup_{g \in \mathcal{G}} |g'(0)| = 1$,

$$M_2 = \sup_{f \in \mathcal{F}} |f'(\frac{1}{3})|$$

$$= \sup_{g \in \mathcal{G}} |(g \circ \phi_{\frac{1}{3}})'(\frac{1}{3})|$$

$$= \sup_{g \in \mathcal{G}} |g'(\phi_{\frac{1}{3}}(\frac{1}{3}))\phi_{\frac{1}{3}}'(\frac{1}{3})|$$

$$= \frac{9}{8} \sup_{g \in \mathcal{G}} |g'(0)|$$

$$= \frac{9}{8}.$$ 

**Comment:** Avoid the following common mistake: noting that $M_1 \leq \frac{3}{7}$ and $M_2 \leq \frac{9}{8}$, and then erroneously arguing that since $M_1$ and $M_2$ are attained, $M_1 = \frac{3}{7}$ and $M_2 = \frac{9}{8}$. 

3
4. Show that $e^z = 2z + 1$ for exactly one $z \in \mathbb{D}$.

**Solution:** Let $f(z) = 2z$ and $g(z) = 2z + 1 - e^z$. If $|z| = 1$, then

$$|f(z) - g(z)| = |e^z - 1|$$

$$= |z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots|$$

$$\leq 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

$$= e - 1$$

$$< 2$$

$$= |f(z)|.$$

Therefore, as $f$ has exactly one zero in $\mathbb{D}$, by Rouche’s Theorem, $g$ has exactly one zero in $\mathbb{D}$.

5. Prove or disprove: every ideal in the ring of entire functions is principal.

**Solution:** Let $\{a_k\}$ be a sequence of distinct points in $\mathbb{C}$ with $a_k \to \infty$ as $k \to \infty$. Choose a sequence of positive integers, $\{p_k\}$, such that the function $f$ defined by the product,

$$f(z) = \prod_{k=1}^{\infty} E_{p_k} \left( \frac{z}{a_k} \right)$$

has zeros only at the points $a_k$. For each $n \geq 1$, define $f_n$ by

$$f_n(z) = \prod_{k=n}^{\infty} E_{p_k} \left( \frac{z}{a_k} \right).$$

Thus, if $n \geq 1$, $f_n$ has 0’s exactly at the points, $a_k$, $k \geq n$. Let $I$ be the ideal generated in $H(\mathbb{D})$ by the functions $f_n$, $n \geq 1$.

We claim that $I$ is not a principal ideal. To prove this claim, we argue by contradiction. Accordingly, suppose that $g \in I$ generates $I$. As $g \in I$, there exist integers $k_1 < k_2 < \ldots < k_m$ and functions, $h_1, h_2, \ldots, h_m \in H(\mathbb{C})$ such that

$$g = h_1 f_{k_1} + h_2 f_{k_2} + \ldots + h_m f_{k_m}.$$
Hence, \( g(a_k) = 0 \) if \( k \geq k_m \). As \( I \) is generated by \( g \), it follows that if \( h \in I \), then \( h(a_k) = 0 \) if \( k \geq k_m \). But \( f_{k_m+1} \in I \) but \( f_{k_m+1}(a_{k_m}) \neq 0 \).

**Comment:** The above reasoning can be adopted to work with any sequence of functions \( \{f_n\} \) provided for each \( n \), \( f_n \) has an infinite number of simple 0’s and the 0-set of \( f_{n+1} \) is a proper subset of the 0-set of \( f_n \). In the solution above, such a sequence was constructed by employing a Weierstrass product. Other strategies are possible. For example, one could take \( f_n(z) = \sin(2^{-n}z) \).

6. Recall the function \( \Gamma \), defined by the formula,

\[
\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}},
\]

where \( \gamma \) is chosen so that \( \Gamma(1) = 1 \).

(i) Show that \( \gamma = \lim_{n \to \infty} (1 + 2 + \ldots + \frac{1}{n} - \log n) \)

(ii) Show that \( \Gamma'(1) = -\gamma \).

**Solution:**

(i) Evidently, as \( \gamma \) is chosen so that \( \Gamma(1) = 1 \),

\[
e^\gamma = \prod_{n=1}^{\infty} (1 + \frac{1}{n})^{-1} e^{\frac{1}{n}}.
\]
Hence,

\[
\gamma = \sum_{n=1}^{\infty} \left[ \log \left( \frac{1}{n} \right)^{-1} + \frac{1}{n} \right]
\]

\[
= \lim_{n \to \infty} \sum_{k=1}^{n} \left[ \log \frac{k}{k+1} + \frac{1}{k} \right]
\]

\[
= \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n} (\log k - \log(k+1)) \right)
\]

\[
= \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) \right)
\]

\[
= \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log(n) \right).
\]

(ii) Logarithmic differentiation yields that

\[
\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left( -\frac{1}{1 + \frac{z}{n}} + \frac{1}{n} \right)
\]

\[
= -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(n + z)},
\]

with uniform convergence on compact subsets of \( \mathbb{C} \setminus \{0, -1, -2, \ldots\} \). Setting \( z = 1 \) gives that

\[
\Gamma'(1) = \frac{\Gamma'(1)}{\Gamma(1)} = -\gamma - 1 + \sum_{n=1}^{\infty} \frac{1}{n(n + 1)} = -\gamma.
\]

7. Recall the function \( \zeta \), defined by the series,

\[
\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.
\]

Show that this series converges uniformly on compact subsets of \( \text{Re} \ z > 1 \).
Solution: Fix a compact set $K \subseteq \{z \mid \text{Re } z > 1\}$. Let $p = \min_{z \in K} \text{Re } z$. As $K$ is compact, $p > 1$. Also, if $z \in K$,
\[ \left| \frac{1}{n^z} \right| = \frac{1}{n^{\text{Re } z}} \leq \frac{1}{n^p}. \]
Furthermore, as $p > 1$, the series $\sum_n \frac{1}{n^p}$ converges. Therefore, by the Weierstrass M-test, the series $\sum_n \frac{1}{n^p}$ converges uniformly on $K$.

8. Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function satisfying $f(0) = 0$. Prove that
\[ |f(z) + f(-z)| \leq 2|z|^2 \]
for all $z \in \mathbb{D}$. Further, show that this inequality is strict for all $z \in \mathbb{D} \setminus \{0\}$ unless $f(z) + f(-z) = 2cz^2$ for some $c \in \mathbb{C}$ with $|c| = 1$.

Solution 1: Since $f(0) = 0$, we may define an analytic function $g$ on $\mathbb{D}$ by the formula,
\[ g(z) = \frac{f(z) + f(-z)}{2z}. \]
I claim that $g(0) = 0$ and $\sup_{z \in \mathbb{D}} |g(z)| \leq 1$.

To see that $g(0) = 0$, note that
\[
g(0) = \lim_{z \to 0} g(z)
= \lim_{z \to 0} \frac{f(z) + f(-z)}{2z}
= \frac{1}{2} \lim_{z \to 0} \frac{f(z) - f(0)}{z} - \frac{1}{2} \lim_{z \to 0} \frac{f(-z) - f(0)}{-z}
= \frac{1}{2} f'(0) - \frac{1}{2} f'(0)
= 0.
\]
(Alternately, use power series or simply observe that $g$ is an odd function.)
To see that $\sup_{z \in \mathbb{D}} |g(z)| \leq 1$ note that if $z \in \mathbb{D}$ and $|z| \leq r < 1$, then

$$|g(z)| \leq \max_{|w| \leq r} |g(w)|$$

(maximum principle) $= \max_{|w|=r} |g(w)|$

$$= \max_{|w|=r} |\frac{f(w) + f(-w)}{2w}|$$

$(f : \mathbb{D} \to \mathbb{D}) \leq \frac{1}{r}.$

Thus, as this inequality holds for all $r \in [|z|, 1)$, $|g(z)| \leq 1$.

An alternate way to show that $|g| \leq 1$ on $\mathbb{D}$ is to note that $f$ satisfies the hypotheses of Schwarz’s Lemma so that $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$. This implies that the function defined by $k(z) = f(z)/z$ for $z \in \mathbb{D} \setminus \{0\}$ and $h(0) = 0$ is analytic on $\mathbb{D}$ and satisfies the inequality $|k(z)| \leq 1$ on $\mathbb{D}$. Thus, if $z \in \mathbb{D},$

$$|g(z)| = \left| \frac{f(z) + f(-z)}{2z} \right| = \left| \frac{1}{2} k(z) - \frac{1}{2} k(-z) \right| \leq \frac{1}{2} |k(z)| + \frac{1}{2} |k(-z)| \leq 1.$$

As $g(0) = 0$ and $\sup_{z \in \mathbb{D}} |g(z)| \leq 1$, Schwarz’s Lemma implies that $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$. This unravels to $|f(z) + f(-z)| \leq 2|z|^2$ for all $z \in \mathbb{D}$. If $z_0 \in \mathbb{D} \setminus \{0\}$ and $|f(z_0) + f(-z_0)| = 2|z_0|^2$, then $g(z_0) = |z_0|$ and Schwarz’s Lemma implies that there exists a $c$ with $|c| = 1$ such that $g(z) = cz$ for all $z \in \mathbb{D}$. This implies that $f(z) + f(-z) = 2cz^2$ for all $z \in \mathbb{D}$.

**Solution 2:** Let a function $h$ be defined on $\mathbb{D}$ by the formula

$$h(z^2) = \frac{f(z) + f(-z)}{2}.$$

As $(f(z) + f(-z))/2$ is an even analytic function, $h$ is both well defined and analytic on $\mathbb{D}$. Note that since $f(0) = 0$, so also $h(0) = 0$. Furthermore, as $|f| \leq 1$ on $\mathbb{D}$, so also $|h| \leq 1$ on $\mathbb{D}$. We have shown that that $h$ satisfies the hypotheses of the Schwarz Lemma. Hence, $|h(z^2)| \leq |z^2|$ for all $z \in \mathbb{D}$. Equivalently, $|f(z) + f(-z)| \leq 2|z|^2$ for all $z \in \mathbb{D}$.

If $z_0 \in \mathbb{D} \setminus \{0\}$ and $|f(z_0) + f(-z_0)| = 2|z_0|^2$, then $h(z_0^2) = |z_0|^2$, and Schwarz’s Lemma implies that there exists a $c$ with $|c| = 1$ such that $h(z) = cz$ for all $z \in \mathbb{D}$. This implies that $f(z) + f(-z) = 2cz^2$ for all $z \in \mathbb{D}$.