Math 109 Homework 1 Solutions

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1. Prove that $\{x \in \mathbb{R} \,|\, [0,1] \subseteq [0,x)\} = (1,\infty)$.

Solution. We employ the Set Equality Principle. We first prove that

$$\{x \in \mathbb{R} \,|\, [0,1] \subseteq [0,x)\} \subseteq (1,\infty). \tag{1}$$

If $u \in \{x \in \mathbb{R} \,|\, [0,1] \subseteq [0,x)\}$, then $[0,1] \subseteq [0,u)$ (by the definition of $\{x \in \mathbb{R} \,|\, [0,1] \subseteq [0,x)\}$). Since $1 \in [0,1]$, it follows that $1 \in [0,u)$ (by the definition of set inclusion). Hence, $1 < u$ (by the definition of [0,u)). Consequently, $u \in (1,\infty)$ (by the definition of $(1,\infty)$). This proves (1).

We next prove that

$$(1,\infty) \subseteq \{x \in \mathbb{R} \,|\, [0,1] \subseteq [0,x)\}. \tag{2}$$

Accordingly, let $u \in (1,\infty)$ so that $1 < u$ (by the definition of $(1,\infty)$) . (2) will follow if we can show that $u \in \{x \in \mathbb{R} \,|\, [0,1] \subseteq [0,x)\}$ (by the definition of set inclusion). But, (by the definition of $\{x \in \mathbb{R} \,|\, [0,1] \subseteq [0,x)\}$), this is equivalent to showing that

$$[0,1] \subseteq [0,u). \tag{3}$$

Let $v \in [0,1]$ so that $v \in \mathbb{R}$ and $0 \leq v \leq 1$. Since $v \leq 1$ and $1 < u$, it follows that $v < u$. Thus, $v \in \mathbb{R}$ and $0 \leq v < u$, or equivalently, $v \in [0,u)$. This proves (3).

2. Let A, B, C, D be sets. Use the Equality Principle to prove that

$$(A \cup B) \cap (C \cup D) = (A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D). \tag{4}$$

Solution.

$\subseteq$: If $x \in (A \cup B) \cap (C \cup D)$, then $x \in A \cup B$ and $x \in C \cup D$. Since $x \in A \cup B$, either $x \in A$ or $x \in B$. Likewise, since $x \in C \cup D$, either $x \in C$ or $x \in D$. Thus, at least one of the following four possibilities must occur:

1. $x \in A$ and $x \in C$,
2. $x \in A$ and $x \in D$,
3. $x \in B$ and $x \in C$,
4. $x \in B$ and $x \in D$.

If 1. holds, then $x \in A \cap C$. Likewise, 2. implies $x \in A \cap D$, 3. implies $x \in B \cap C$, and 4. implies $x \in B \cap D$. Since each of the sets $(A \cap C)$, $(A \cap D)$, $B \cap C$, and $B \cap D$ is a subset of $(A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D)$, it follows that $x \in (A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D)$.

$\supseteq$: If $x \in (A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D)$, then x must be an element of at least one of the sets $(A \cap C)$, $(A \cap D)$, $B \cap C$, and $B \cap D$. If x is an element of $(A \cap C)$, the first of these sets, then $x \in A$ and $x \in C$. Since $x \in A$, $x \in A \cup B$ and since $x \in C$, $x \in C \cup D$. Hence, $x \in (A \cup B) \cap (C \cup D)$. Likewise, in the other three cases when x is an element of one of the sets $(A \cap D)$, $B \cap C$, and $B \cap D$, $x \in x \in (A \cup B) \cap (C \cup D)$.

3. Give a proof of (4) using Proposition 1.34 on page 17.

Solution

$$\begin{aligned}(A \cup B) \cap (C \cup D) &= \Big(A \cap (C \cup D)\Big) \cup \Big(B \cap (C \cup D)\Big) \\ &= \Big((A \cap C) \cup (A \cap D)\Big) \cup \Big((B \cap C) \cup (B \cap D)\Big) \\ &= (A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D).\end{aligned}$$

4. Let $\mathbb{R}^+ = \{\delta \in \mathbb{R} \mid \delta > 0\}$. Compute each of the following sets. Be sure to justify your answer.

$$(i) \qquad \bigcup_{\delta \in \mathbb{R}+} (-\delta, \delta) \text{ and } \bigcap_{\delta \in \mathbb{R}+} (-\delta, \delta)$$

$$(ii) \qquad \bigcup_{\delta\in\mathbb{R}+}[\delta,\infty) \text{ and } \bigcap_{\delta\in\mathbb{R}+}[\delta,\infty).$$

Solution.

(i) $\cup_{\delta\in\mathbb{R}+}(-\delta,\delta)=\mathbb{R}$ and $\cap_{\delta\in\mathbb{R}+}(-\delta,\delta)=\{0\}$.

To see that $\cup_{\delta\in\mathbb{R}+}(-\delta,\delta)=\mathbb{R}$, note that if $x\in\cup_{\delta\in\mathbb{R}+}(-\delta,\delta)$, then $x\in(-\delta,\delta)$ for some $\delta\in\mathbb{R}^+$. In particular, it follows by the definition of $(-\delta,\delta)$ that $x\in\mathbb{R}$. On the other hand, if $x\in\mathbb{R}$, then $-(|x|+1)<x<|x|+1$, i.e., $x\in(-(|x|+1),|x|+1)$. Since $|x|+1\in\mathbb{R}^+$, it follows that $x\in\cup_{\delta\in\mathbb{R}+}(-\delta,\delta)$.

To see that $\cap_{\delta\in\mathbb{R}+}(-\delta,\delta)=\{0\}$, note first that if $x\in\cap_{\delta\in\mathbb{R}+}(-\delta,\delta)$, then $x\in(-\delta,\delta)$ for every $\delta\in\mathbb{R}^+$. Equivalently, $x\in\mathbb{R}$ and $-\delta<x<\delta$ for every $\delta\in\mathbb{R}$ such that $\delta>0$. Since $x<\delta$ whenever $\delta>0$, $x\leq 0$. Likewise, since $-\delta<x$ whenever $\delta>0$, $x\geq 0$. Since we have that both $x\leq 0$ and $x\geq 0$, $x=0$. Now assume that $x\in\{0\}$ (i.e. $x=0$). Then since $-\delta<0<\delta$ whenever $\delta>0$, $0\in(-\delta,\delta)$ for every $\delta\in\mathbb{R}^+$. Therefore, $0\in\cap_{\delta\in\mathbb{R}+}(-\delta,\delta)$.

(ii) $\cup_{\delta\in\mathbb{R}+}[\delta,\infty)=\mathbb{R}^+$ and $\cap_{\delta\in\mathbb{R}+}[\delta,\infty)=\varnothing$.

If $x\in\cup_{\delta\in\mathbb{R}+}[\delta,\infty)$ then $x\in[\delta,\infty)$ for some $\delta\in\mathbb{R}^+$. But then, $0<\delta\leq x$, i.e., $x\in\mathbb{R}^+$. Conversely, if $x\in\mathbb{R}^+$, then, as $x\in[x,\infty)$, $x\in\cup_{\delta\in\mathbb{R}+}[\delta,\infty)$.

If x were in $\cap_{\delta\in\mathbb{R}+}[\delta,\infty)$, then as $|x|+1\in\mathbb{R}^+$, it would have to be the case that $x\in[|x|+1,\infty)$. This would imply that $|x|+1<x$, which is clearly absurd. Therefore, $\cap_{\delta\in\mathbb{R}+}[\delta,\infty)=\varnothing$.