1. Give the definitions for the following notions.
(i) \([a, b]\), where \(a, b \in \mathbb{R}\);
(ii) \(g \circ f\), where \(f : X \to Y\) and \(g : Y \to Z\) are functions;
(iii) \(f^* (A)\), where \(f : X \to Y\) is a function and \(A \subseteq X\);
(iv) \(\text{Graph}(f)\), where \(f : X \to Y\) is a function.

Solution.
(i) \([a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}\).
(ii) \(g \circ f : X \to Z\) and \((g \circ f)(x) = g(f(x))\) for \(x \in X\).
(iii) \(f^* (A) = \{ f(x) \mid x \in A \}\), or equivalently, \(f^* (A) = \{ y \mid \exists x \in A \ y = f(x) \}\).
(iv) \(\text{Graph}(f) = \{ (x, f(x)) \mid x \in X \}\), or equivalently, \(\text{Graph}(f) = \{ (x, y) \in X \times Y \mid \exists x \in X \ y = f(x) \}\).

2. Let \(X\) and \(Y\) be sets and let \(\{ Y_i \}_{i \in I}\) be a collection of subsets of \(Y\). Prove that if \(f : X \to Y\) is a function, then
\[
\bigcup_{i \in I} f^* (Y_i) = \bigcup_{i \in I} f^* (Y_i).
\]

Solution The assertion is meaningless. If \(f : X \to Y\), then \(f^* : P(X) \to P(Y)\). Since \(Y_i\) and \(\bigcup_{i \in I} Y_i\) are subsets of \(Y\), as opposed to \(X\), the expressions \(f^* (Y_i)\) and \(f^* (\bigcup_{i \in I} Y_i)\) do not make sense.
One way to fix the problem would be to replace the collection \( \{ Y_\i \}_{\i \in I} \) of subsets of \( Y \) with a collection \( \{ X_\i \}_{\i \in I} \) of subsets of \( X \). We then would have that

\[
y \in f_*(\bigcup_{\i \in I} X_\i)
\iff y = f(x) \text{ for some } x \in \bigcup_{\i \in I} X_\i
\iff \text{ for some } \i \in I, y = f(x) \text{ for some } x \in X_\i
\iff \text{ for some } \i \in I, y \in f_*(X_\i)
\iff y \in \bigcup_{\i \in I} f_*(X_\i).
\]

which proves that

\[
f_*(\bigcup_{\i \in I} X_\i) = \bigcup_{\i \in I} f_*(X_\i).
\]

**Remark:** The problem could also be fixed by leaving the the collection \( \{ Y_\i \}_{\i \in I} \) of subsets of \( Y \) unchanged, but replacing \( f_* \) with \( f^* \). For then it is the case that

\[
f^*(\bigcup_{\i \in I} Y_\i) = \bigcup_{\i \in I} f^*(Y_\i).
\]

3. Let \( X, Y, \) and \( Z \) be sets and let \( f : X \to Y \) and \( g : Y \to Z \) be functions.

(i) Prove that if \( g \circ f \) is 1-1, then \( f \) is 1-1.

(ii) Prove that if \( g \circ f \) is onto, then \( g \) is onto.

**Solution.** (cf. the solution to Quiz 4).

(i) Assume that \( g \circ f \) is 1-1. To see that \( f \) is 1-1, let \( x_1, x_2 \in X \) with \( f(x_1) = f(x_2) \). Then

\[
(g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2).
\]
Consequently, since $g \circ f$ an 1-1, it follows that $x_1 = x_2$.

(ii) Assume that $g \circ f$ is onto. To see that $g$ is onto, fix $z \in Z$. Since $g \circ f$ is onto, there exists $x \in X$ such that $z = (g \circ f)(x)$. Let $y = f(x)$. Then $y \in Y$ and
\[
z = (g \circ f)(x) = g(f(x)) = g(y).
\]

4. Let $X$ and $Y$ be sets and let $S \subseteq X \times Y$. Find and prove a necessary and sufficient condition for $S$ to be the graph of a function.

Solution. Let $X$ and $Y$ be sets and let $S \subseteq X \times Y$. $S$ is the graph of a function if and only if
\[
\forall x \in X \exists y \in Y \quad (x, y) \in S
\]
and
\[
\forall x \in X \forall y_1 \in Y \forall y_2 \in Y \quad (x, y_1), (x, y_2) \in S \implies y_1 = y_2.
\]

If $S$ satisfies (1) and (2), then for each $x \in X$ there exists a unique $y \in Y$ such that $(x, y) \in S$. For each $x \in X$ let $f(x) = y$ where $y$ is the unique $y \in Y$ such that $(x, y) \in S$. With this definition of $f$, for all $x \in X$ and all $y \in Y$,
\[
y = f(x) \iff (x, y) \in S,
\]
i.e., $S = \text{Graph}(f)$.

Now let $f : X \to Y$ be a function and let $S = \text{Graph}(f)$. Then (1) holds because $f(x)$ is defined for all $x \in X$ and (2) holds because for each $x \in X$ there is a unique $y \in Y$ such that $y = f(x)$.

5. Use Mathematical Induction to prove the following statement.

\[
\forall n \in \mathbb{N} \quad \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n \cdot (n + 1)} = \frac{n}{n + 1}.
\]
Solution. We wish to prove that
\[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1} \]  
for all \( n \in \mathbb{N} \). First observe that
\[ \frac{1}{1 \cdot 2} = \frac{1}{1+1}, \]
i.e., (3) holds when \( n = 1 \).

Now assume that \( n \in \mathbb{N} \) and (3) holds. Then
\[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} \]
\[ = \left( \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} \right) + \frac{1}{(n+1) \cdot (n+2)} \]
\[ = \frac{n}{n+1} + \frac{1}{(n+1) \cdot (n+2)} \]
\[ = \frac{n(n+2)}{(n+1) \cdot (n+2)} + \frac{1}{(n+1) \cdot (n+2)} \]
\[ = \frac{(n+1)^2}{(n+1) \cdot (n+2)} \]
\[ = \frac{n+1}{n+2}, \]
i.e., (3) holds when \( n \) is replaced with \( n + 1 \).

Summarizing, we have shown that (3) holds when \( n = 1 \). We have also shown that if (3) holds for a given \( n \in \mathbb{N} \), then (3) holds when \( n \) is replaced with \( n + 1 \). Therefore, by the Principle of Mathematical Induction, (3) holds for all \( n \in \mathbb{N} \).
Remark. This problem can also be solved in the following way. One exploits the fact that the sum is ‘telescoping’.

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n \cdot (n + 1)}
\]

\[
= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \ldots + (\frac{1}{n} - \frac{1}{n + 1})
\]

\[
= 1 - \frac{1}{n + 1}
\]

\[
= \frac{n}{n + 1}.
\]

However, as you are asked to prove the assertion using the Principle of Mathematical Induction, this approach to the problem is not legal.