Problem. Let $A$, $B$, and $C$ be sets. Show that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Solution. By the set equality principle we need to show that

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C) \quad (1)$$

and

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C). \quad (2)$$

To prove (1) assume that $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, so that $x \in (A \cup B) \cap (A \cup C)$. On the other hand, if $x \in B \cap C$, then $x \in B$ and $x \in C$. Hence, $x \in A \cup B$ and $x \in A \cup C$, and again we have that $x \in (A \cup B) \cap (A \cup C)$.

To prove (2) assume that $x \in (A \cup B) \cap (A \cup C)$, so that $x \in A \cup B$ and $x \in A \cup C$. Now, either $x \in A$ or $x \notin A$. If $x \in A$, then $x \in A \cup (B \cap C)$. On the other hand, if $x \notin A$, then, as $x \in A \cup B$ and $x \in A \cup C$, necessarily, $x \in B$ and $x \in C$. Therefore, $x \in B \cap C$ and we see that, again, it is the case that $x \in A \cup (B \cap C)$. 
