Problem. Let $X$, $Y$, and $Z$ be sets and let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove the following assertions.

(i) If $g \circ f$ is an injection, then $f$ is an injection.

(ii) If $g \circ f$ is a surjection, then $g$ is a surjection.

Solution.

(i) Assume that $g \circ f$ is an injection. If $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$, then

$$(g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2).$$

Consequently, since $g \circ f$ an injection, it follows that $x_1 = x_2$.

Summarizing, we have shown that if $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$, then $x_1 = x_2$. Therefore, $f$ is injective.

(ii) Assume that $g \circ f$ is a surjection. Fix $z \in Z$. Since $g \circ f$ is a surjection, there exists $x \in X$ such that $z = (g \circ f)(x)$. Let $y = f(x)$. Then $y \in Y$ and

$$z = (g \circ f)(x) = g(f(x)) = g(y).$$

Summarizing, we have shown that if $z \in Z$, then there exists $y \in Y$ such that $z = g(y)$. Therefore, $g$ is surjective.