Some Remarks on Induction

Jim Agler

There is a second principle of induction, often referred to as the *Strong Principle of Mathematical Induction*. With strong induction, to prove that \( P(n) \) is true for all \( n \), it suffices to prove the following two facts for some \( m \in \mathbb{N} \):

\[
P(1), P(2), \ldots, P(m) \text{ are true} \quad (1)
\]

and

\[
\text{if } m \geq n \in \mathbb{N} \text{ and } P(k) \text{ is true for each } k \in \{1, 2, \ldots, n\}, \text{ then } P(n+1) \text{ is true}. \quad (2)
\]

Strong induction can be used in cases where it is difficult to prove that \( P(n+1) \) is true from the single assumption that \( P(n) \) is true, and yet it is relatively easy to prove that \( P(n+1) \) is true from the \( n \) assumptions that \( P(1), P(2), \ldots, P(n) \) are all true.

**Example 1.** Prove that \( \forall n \in \mathbb{N} \ F_n \geq \left( \frac{3}{2}\right)^{n-2} \) using strong induction.

**Solution.** We use strong induction with \( m = 2 \). Note that

\[
F_1 = 1 \geq \frac{2}{3} = \left( \frac{3}{2} \right)^{-1} \text{ and } F_2 = 1 = \left( \frac{3}{2} \right)^0
\]

so that the assertion is true when \( n = 1 \) and \( n = 2 \).

Now assume that \( n \geq 2 \) and \( F_k \geq \left( \frac{3}{2} \right)^{k} \) for all \( k \in \{1, 2, \ldots, n\} \). Then

\[
F_{n+1} = F_{n-1} + F_n \\
\geq \left( \frac{3}{2} \right)^{n-3} + \left( \frac{3}{2} \right)^{n-2} \\
= \left( \frac{3}{2} \right)^{n-3} \left( 1 + \frac{3}{2} \right) \\
= \left( \frac{3}{2} \right)^{n-3} \frac{5}{2} \\
\geq \left( \frac{3}{2} \right)^{n-3} \left( \frac{3}{2} \right)^2 \\
= \left( \frac{3}{2} \right)^{n-1}. \quad \square
\]

**Remark.** Note that the choice \( m = 2 \) in the above example was due to the fact that \( F_{n+1} \) is defined in terms of the previous two Fibonacci numbers.

It is never necessary to use strong induction. One can also prove the assertion above using normal mathematical induction provided one chooses one’s predicate wisely.

**Example 2.** Prove that \( \forall n \in \mathbb{N} \ F_n \geq \left( \frac{3}{2}\right)^{n-2} \) using regular induction.

Let \( P(n) \) be the predicate

\[
F_k \geq \left( \frac{3}{2} \right)^{k-2} \text{ whenever } k \in \{1, 2, \ldots, n\}.
\]

Clearly, \( F_n \geq \left( \frac{3}{2}\right)^{n-2} \) for all \( n \in \mathbb{N} \) will follow if \( P(n) \) is true for all \( n \in \mathbb{N} \) such that \( n \geq 2 \). We prove this latter assertion by mathematical induction. Notice first that

\[
F_1 = 1 \geq \frac{2}{3} = \left( \frac{3}{2} \right)^{-1} \text{ and } F_2 = 1 = \left( \frac{3}{2} \right)^0
\]

Therefore, \( P(n) \) is true in the base case when \( n = 2 \).
Now assume that \( n \in \mathbb{N}, n \geq 2 \) and \( P(n) \) is true. We wish to show that \( P(n + 1) \) is true, i.e., that \( F_k \geq (\frac{3}{2})^{k-2} \) for all \( k \in \{1, 2, \ldots, n+1\} \). But since \( P(n) \) is true, \( F_k \geq (\frac{3}{2})^{k-2} \) for all \( k \in \{1, 2, \ldots, n\} \). There remains to show that \( F_{n+1} \geq (\frac{3}{2})^{n-1} \). But

\[
F_{n+1} = F_{n-1} + F_n \\
\geq (\frac{3}{2})^{n-3} + (\frac{3}{2})^{n-2} \\
= (\frac{3}{2})^{n-3}(1 + \frac{3}{2}) \\
= (\frac{3}{2})^{n-3} \cdot \frac{5}{2} \\
\geq (\frac{3}{2})^{n-3} \left( \frac{3}{2} \right)^2 \\
= (\frac{3}{2})^{n-1}. \quad \square
\]

It is also possible to prove that \( \forall n \in \mathbb{N} \) \( F_n \geq (\frac{3}{2})^{n-2} \) avoiding induction altogether.

**Example 3.** Prove that \( \forall n \in \mathbb{N} \) \( F_n \geq (\frac{3}{2})^{n-2} \) using the Well Ordering Principle.

**Solution.** Let

\[
S = \{n \in \mathbb{N} | F_n < (\frac{3}{2})^{n-2}\},
\]

so that we wish to prove that \( S = \emptyset \). We argue by contradiction. If \( S \neq \emptyset \), the by the Well Ordering Principle, \( S \) has a least element \( m \). Since \( m \in S \),

\[
F_m < (\frac{3}{2})^{m-2}. \tag{3}
\]

Therefore, as \( F_1 = 1 \geq \frac{2}{3} = (\frac{3}{2})^{-1} \) and \( F_2 = 1 = (\frac{3}{2})^0 \), \( m \geq 3 \). Therefore, if we let \( n = m - 1 \), then \( n - 1, n \in \mathbb{N} \) and (as \( m \) is the least element of \( S \)) that both

\[
F_{n-1} \geq (\frac{3}{2})^{n-3} \text{ and } F_n \geq (\frac{3}{2})^{n-2}.
\]

But then

\[
F_m = F_{n+1} = F_{n-1} + F_n \\
\geq (\frac{3}{2})^{n-3} + (\frac{3}{2})^{n-2} \\
= (\frac{3}{2})^{n-3}(1 + \frac{3}{2}) \\
= (\frac{3}{2})^{n-3} \cdot \frac{5}{2} \\
\geq (\frac{3}{2})^{n-3} \left( \frac{3}{2} \right)^2 \\
= (\frac{3}{2})^{n-1} \\
= (\frac{3}{2})^{m-2} \quad \square
\]

contradicting (3).

**Remark.** We have given three different proofs that \( \forall n \in \mathbb{N} \) \( F_n \geq (\frac{3}{2})^{n-2} \), one using strong induction, one using regular induction, and one using the Well Ordering Principle. There is no valid mathematical reason to choose one of these approaches over the other. Rather, which approach is adopted is entirely a matter of the personal tastes of the writer of the proof (or the personal tastes of your professor should he choose to force them on you).