Math 120 Practice Final Exam Solutions

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1. Use Euler’s formula to derive the trigonometric identities

\[ \cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta \quad \text{and} \quad \sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta. \]

**Solution.** Using Euler’s formula twice and the identity,

\[ (z + w)^3 = z^3 + 3z^2w + 3zw^2 + w^3, \]

we have that

\[
\begin{align*}
\cos 3\theta + i \sin 3\theta &= e^{i(3\theta)} \\
&= (e^{i\theta})^3 \\
&= (\cos \theta + i \sin \theta)^3 \\
&= (\cos \theta)^3 + 3(\cos \theta)^2(i \sin \theta) + 3(\cos \theta)(i \sin \theta)^2 + (i \sin \theta)^3 \\
&= (\cos^3 \theta - 3 \cos \theta \sin^2 \theta) + i(3 \cos^2 \theta \sin \theta - \sin^3 \theta).
\end{align*}
\]

The identities follow by equating real and imaginary parts.

2. Show that if \( f \) is entire, \( f(0) = 1 \), and \( f'(z) = 2f(z) \) for all \( z \in \mathbb{C} \), then \( f(z) = e^{2z} \) for all \( z \in \mathbb{C} \).

**Solution.** Suppose \( f \) is entire, \( f(0) = 1 \), and \( f'(z) = 2f(z) \) for all \( z \in \mathbb{C} \). Define a function by the formula

\[ g(z) = \frac{f(z)}{e^{2z}}, \quad z \in \mathbb{C}. \]

Since \( e^{2z} \) is entire, \( e^{2z} \neq 0 \) for all \( z \in \mathbb{C} \), and \( f(z) \) is assumed to be entire, it follows
that $g$ is entire. Also, since $f'(z) = 2f(z)$, we have by the quotient rule that,

$$g'(z) = \frac{d}{dz} \frac{f(z)}{e^{2z}}$$

$$= \frac{f'(z)e^{2z} - f(z)(2e^{2z})}{(e^{2z})^2}$$

$$= \frac{2f(z)e^{2z} - f(z)(2e^{2z})}{(e^{2z})^2}$$

$$= 0$$

for all $z \in \mathbb{C}$. Since $g$ is entire and $g'(z) = 0$ for all $z \in \mathbb{C}$ it follows that there exists a constant $c$ such that $g(z) = c$ for all $z \in \mathbb{C}$ (cf. Theorem pg. 73 in the text). But since $f(0) = 1$, so also $g(0) = 1$ so that necessarily, $c = 1$. Therefore,

$$\frac{f(z)}{e^{2z}} = g(z) = 1$$

for all $z \in \mathbb{C}$. I follows that $f(z) = e^{2z}$ for all $z \in \mathbb{C}$, as was to be proved.

3. Let $f(z)$ be defined by the formula

$$f(z) = e^{x^2-y^2} \cos(xy) + i \ e^{x^2-y^2} \sin(xy), \quad z = x + iy.$$  

Show that $f$ is entire and that $f'(z) = zf(z)$ for all $z$.

Solution 1. $f = u + iv$ where

$$u(x, y) = e^{x^2-y^2} \cos(xy) \quad \text{and} \quad v(x, y) = e^{x^2-y^2} \sin(xy).$$

Using the product rule we have that

$$u_x = x \ e^{x^2-y^2} \cos(xy) - y \ e^{x^2-y^2} \sin(xy),$$

$$u_y = -y \ e^{x^2-y^2} \cos(xy) - x \ e^{x^2-y^2} \sin(xy),$$

$$v_x = x \ e^{x^2-y^2} \sin(xy) + y \ e^{x^2-y^2} \cos(xy), \quad \text{and}$$

$$v_y = -y \ e^{x^2-y^2} \sin(xy) + x \ e^{x^2-y^2} \cos(xy).$$
Noting that the Cauchy-Riemann equations,

\[ u_x = v_y \quad \text{and} \quad u_y = -v_x, \]

are satisfied, we conclude that \( f \) is entire (cf. Theorem pg. 66). Furthermore,

\[ f'(z) = u_x + iv_x \]

\[ = \left( x e^{\frac{x^2-y^2}{2}} \cos(xy) - y e^{\frac{x^2-y^2}{2}} \sin(xy) \right) + i \left( x e^{\frac{x^2-y^2}{2}} \sin(xy) + y e^{\frac{x^2-y^2}{2}} \cos(xy) \right) \]

\[ = (x + iy) \left( e^{\frac{x^2-y^2}{2}} \cos(xy) + i e^{\frac{x^2-y^2}{2}} \sin(xy) \right) \]

\[ = zf(z). \]

**Solution 2.** If \( w = u + iv \), then

\[ e^w = e^u (\cos v + i \sin v). \]

Also, if \( z = x + iy \), then

\[ \frac{1}{2} z^2 = \frac{x^2-y^2}{2} + ixy. \]

Hence,

\[ e^{\frac{1}{2}z^2} = e^{\frac{x^2-y^2}{2}} \cos(xy) + i e^{\frac{x^2-y^2}{2}} \sin(xy), \]

i.e.,

\[ f(z) = e^{\frac{1}{2}z^2}. \]

Since the composition of entire functions is entire, it follows immediately that \( f \) is entire. Also, the Chain Rule gives that

\[ f'(z) = \frac{d}{dz} e^{\frac{1}{2}z^2} = ze^{\frac{1}{2}z^2} = zf(z). \]

4. Give the definitions of \( \sin z \) and \( \cos z \) and prove that

\[ \sin^2 z + \cos^2 z = 1. \] (1)

**Solution 1.** \( \sin z \) and \( \cos z \) are defined for all \( z \in \mathbb{C} \) by the formulas

\[ \sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \]
Therefore,

\[ \sin^2 z + \cos^2 z = \left( \frac{e^{iz} - e^{-iz}}{2i} \right)^2 + \left( \frac{e^{iz} + e^{-iz}}{2} \right)^2 \]

\[ = \left( \frac{e^{2iz} - 2e^{iz}e^{-iz} + e^{-2iz}}{4} \right) + \left( \frac{e^{2iz} + 2e^{iz}e^{-iz} + e^{-2iz}}{4} \right) \]

\[ = \left( \frac{-e^{2iz} + 2 - e^{-2iz}}{4} \right) + \left( \frac{e^{2iz} + 2 + e^{-2iz}}{4} \right) \]

\[ = \frac{4}{4} = 1. \]

**Solution 2.** From their definitions, it is immediate that

\[ \frac{d}{dz} \sin z = \cos z \quad \text{and} \quad \frac{d}{dz} \cos z = -\sin z. \]

Therefore, using the Chain Rule,

\[ \frac{d}{dz} (\sin^2 z + \cos^2 z) = 2 \sin z \cos z + 2 \cos z (-\sin z) = 0 \]

for all \( z \in \mathbb{C} \). Consequently, by the theorem that asserts that an analytic function \( f \) defined on a domain \( D \) satisfying \( f'(z) = 0 \) for all \( z \in D \) must be constant (cf. Section 25 pg. 73), there exists a constant \( c \) such that

\[ \sin^2 z + \cos^2 z = c \quad \text{for all} \quad z \in \mathbb{C}. \]

Letting \( z = 0 \) yields that \( c \) must be equal to 1, which proves (1).

5. Let \( C \) be the boundary of the circular sector, \( 0 \leq r \leq 1, \pi/4 \leq \theta \leq \pi/2 \), the orientation of \( C \) being in the counterclockwise direction. Compute

\[ \int_C |z|^2 \, dz. \]

**Solution.** We break \( C \) up into the following 3 smooth contours:

\[ C_1 : \quad z(t) = te^{i\pi/4}, \quad 0 \leq t \leq 1; \]

\[ C_2 : \quad z(\theta) = e^{i\theta}, \quad \pi/4 \leq \theta \leq \pi/2; \]

\[ C_3 : \quad z(t) = (1-t)i, \quad 0 \leq t \leq 1. \]
We have that

\[
\int_{C_1} |z|^2 \, dz = \int_0^1 |te^{i\pi/4}|^2 e^{i\pi/4} \, dt
\]

\[= e^{i\pi/4} \int_0^1 t^2 \, dt \]

\[= \frac{1}{3} e^{i\pi/4}, \]

\[
\int_{C_2} |z|^2 \, dz = \int_{\pi/4}^{\pi/2} |e^{i\theta}|^2 i e^{i\theta} \, d\theta
\]

\[= \int_{\pi/4}^{\pi/2} i e^{i\theta} \, d\theta \]

\[= e^{i\theta}\bigg|_{\pi/4}^{\pi/2}
\]

\[= i - e^{i\pi/4}, \]

and

\[
\int_{C_3} |z|^2 \, dz = \int_0^1 |(1 - t)i|^2 (-i) \, dt
\]

\[= -i \int_0^1 (1 - t)^2 \, dt \]

\[= -\frac{1}{3} i. \]

Therefore,

\[
\int_{C} |z|^2 \, dz = \int_{C_1} |z|^2 \, dz + \int_{C_2} |z|^2 \, dz + \int_{C_3} |z|^2 \, dz
\]

\[= \frac{1}{3} e^{i\pi/4} + (i - e^{i\pi/4}) + -\frac{1}{3} i
\]

\[= \frac{2}{3} i - \frac{2}{3} e^{i\pi/4}. \]
Remark. Since $e^{i\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, the answer can also be expressed in the form $-\frac{\sqrt{2}}{3} + \frac{2-\sqrt{2}}{3}i$.

6. Let $C$ be an arbitrary smooth contour parametrized by $z = z(t), a \leq t \leq b$. Show directly from the definition of contour integrals that

$$\int_C z \, dz = \frac{z(b)^2}{2} - \frac{z(a)^2}{2}$$

Solution.

$$\int_C z \, dz = \int_a^b z(t) \, z'(t) \, dt$$

$$= \int_a^b \frac{d}{dt} \frac{z(t)^2}{2} \, dt$$

$$= \frac{z(t)^2}{2} \bigg|_a^b$$

$$= \frac{z(b)^2}{2} - \frac{z(a)^2}{2}$$

7. Evaluate

$$\int_C \frac{\cos 3z}{z^2(z - \pi)} \, dz$$

for each of the following contours: (i) $|z| = 2$, (ii) $|z - 2| = 1$, (iii) $|z - 3| = 1$; here, each of these circles is oriented in the counterclockwise direction.

Solution

(i) Notice that 0 is inside $C$. Also, as $\pi$ lies outside $C$, if we set

$$f(z) = \frac{\cos 3z}{z - \pi}$$

then $f$ is analytic on and inside $C$. Therefore, by the Cauchy Integral Formula (with
\( n = 1 \),

\[
\int_C \frac{\cos 3z}{z^2(z - \pi)} \, dz = \int_C \frac{f(z)}{z^2} \, dz
\]

\[
= 2\pi i \left( \frac{1}{2\pi i} \int_C \frac{f(z)}{z^2} \, dz \right)
\]

\[
= 2\pi if'(0).
\]

But

\[
f'(z) = \frac{-3(z - \pi) \sin 3z - \cos 3z}{(z - \pi)^2},
\]

so that

\[
f'(0) = -\frac{1}{\pi^2}.
\]

Therefore,

\[
\int_C \frac{\cos 3z}{z^2(z - \pi)} \, dz = 2\pi i \left( -\frac{1}{\pi^2} \right) = -\frac{2i}{\pi}.
\]

(ii) Since the points 0 and \( \pi \) lie outside \( |z - 2| = 1 \),

\[
\frac{\cos 3z}{z^2(z - \pi)}
\]

is analytic on and inside \( C \). Therefore, by Cauchy’s Theorem,

\[
\int_C \frac{\cos 3z}{z^2(z - \pi)} \, dz = 0.
\]

(iii) Notice that \( \pi \) is inside \( C \). Also, as 0 lies outside \( C \), if we set

\[
g(z) = \frac{\cos 3z}{z^2}
\]

then \( g \) is analytic on and inside \( C \). Therefore, by the Cauchy Integral Formula (with
\[ n = 0, \]
\[
\int_C \frac{\cos 3z}{z^2(z - \pi)} \, dz = \int_C \frac{g(z)}{z - \pi} \, dz
\]
\[
= 2\pi i \left( \frac{1}{2\pi i} \int_C \frac{g(z)}{z} \, dz \right)
\]
\[
= 2\pi i \ g(\pi)
\]
\[
= 2\pi i \left( -\frac{1}{\pi^2} \right)
\]
\[
= -\frac{2i}{\pi}.
\]

8. Let \( f \) be an entire function and assume that \( f \) is bounded away from 0, i.e., there exists an \( \epsilon > 0 \) such that \( |f(z)| \geq \epsilon \) for all \( z \in \mathbb{C} \). Show that \( f \) is constant.

*Solution.* Suppose that \( f \) is an entire function, \( \epsilon > 0 \), and \( |f(z)| \geq \epsilon \) for all \( z \in \mathbb{C} \). Since \( |f(z)| \geq \epsilon \) for all \( z \in \mathbb{C} \), in particular, \( f(z) \neq 0 \) for all \( z \in \mathbb{C} \). Therefore, if we define \( g \) by the formula

\[ g(z) = \frac{1}{f(z)}, \quad z \in \mathbb{C}, \]

then \( g \) is entire. Furthermore, since \( |f(z)| \geq \epsilon \) for all \( z \in \mathbb{C} \),

\[ |g(z)| = \frac{1}{|f(z)|} \leq \frac{1}{\epsilon} \]

for all \( z \in \mathbb{C} \), i.e., \( g \) is bounded. Since \( g \) is a bounded entire function, it follows from Liouville’s Theorem that \( g \) is constant. Since \( g \) is constant, \( f \) is constant.

9. Show that if \( C \) is the circle \( |z| = 3 \), parametrized counterclockwise, and \( t \) is a real number, then

\[ \frac{1}{2\pi i} \int_C \frac{e^{tz}}{z^2 + 1} \, dz = \sin t. \]

*Solution 1.* The Method of Partial Fractions yields the identity

\[ \frac{1}{z^2 + 1} = \frac{1}{2i} \left( \frac{1}{z - i} - \frac{1}{z + i} \right). \]

Therefore, if we let

\[ f(z) = \frac{1}{2i} e^{tz}, \]

\[ 8 \]
then
\[ \frac{e^{tz}}{z^2 + 1} = \frac{f(z)}{z - i} - \frac{f(z)}{z + i}. \]

Hence, by the Cauchy Integral Formula,
\[ \frac{1}{2\pi i} \int_C \frac{e^{tz}}{z^2 + 1} \, dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - i} \, dz - \frac{1}{2\pi i} \int_C \frac{f(z)}{z + i} \, dz \]
\[ = f(i) - f(-i) \]
\[ = \frac{1}{2i} e^{it} - \frac{1}{2i} e^{-it} \]
\[ = \sin t. \]

_Solution 2._ Let \( C^+ \) be the upper half of \( C \) parametrized by \( z(\theta) = 3e^{i\theta}, \ 0 \leq \theta \leq \pi \) and let \( C^- \) be the lower half of \( C \) parametrized by \( z(\theta) = 3e^{i\theta}, \ \pi \leq \theta \leq 2\pi \). Define two closed contours \( \Gamma^+ \) and \( \Gamma^- \) by letting
\[ \Gamma^- = C^+ + [-3, 3] \quad \text{and} \quad \Gamma^- + [3, -3]. \]
Since \([3, -3] = -[-3, 3]\),
\[ \frac{1}{2\pi i} \int_C \frac{e^{tz}}{z^2 + 1} \, dz = \frac{1}{2\pi i} \int_{\Gamma^+} \frac{e^{tz}}{z^2 + 1} \, dz + \frac{1}{2\pi i} \int_{\Gamma^-} \frac{e^{tz}}{z^2 + 1} \, dz. \]
But noting that
\[ z^2 + 1 = (z - i)(z + i), \]
we see using the Cauchy Integral Formula that,
\[ \frac{1}{2\pi i} \int_{\Gamma^+} \frac{e^{tz}}{z^2 + 1} \, dz = \frac{1}{2\pi i} \int_{\Gamma^+} \frac{e^{tz}}{z - i} \, dz = \frac{e^{it}}{2i} \]
and
\[ \frac{1}{2\pi i} \int_{\Gamma^-} \frac{e^{tz}}{z^2 + 1} \, dz = \frac{1}{2\pi i} \int_{\Gamma^-} \frac{e^{tz}}{z + i} \, dz = -\frac{e^{-it}}{2i}. \]
Therefore,
\[ \frac{1}{2\pi i} \int_C \frac{e^{tz}}{z^2 + 1} \, dz = \frac{e^{it}}{2i} - \frac{e^{-it}}{2i} = \sin t. \]

10. Let \( f \) be an entire function satisfying \( f'(0) = 1 \). Show that if \( |f(z)| \leq |z| \) for all \( z \in \mathbb{C} \), then \( f(z) = z \) for all \( z \in \mathbb{C} \).
**Solution 1.** Fix $z_0 \in \mathbb{C}$ and a positive real number $R$. If $z = z_0 + Re^{i\theta}$ is a point on the circle centered at $z_0$ of radius $R$, then

$$|f(z)| \leq |z| = |z_0 + Re^{i\theta}| \leq |z_0| + R.$$ 

Therefore, by Cauchy’s Estimate (with $n = 2$ and $M_R = |z_0| + R$),

$$|f''(z_0)| \leq \frac{2(|z_0| + R)}{R^2}.$$

Since $R$ is an arbitrary positive real number in this inequality, by letting $R \to \infty$ we see that $f''(z_0) = 0$. Since $z_0$ is an arbitrary point in $\mathbb{C}$ it follows that $f''(z) = 0$ for all $z \in \mathbb{C}$. Consequently, there exist constants $a$ and $b$ such that

$$f(z) = a + bz$$

for all $z \in \mathbb{C}$. Now, since we are assuming that $|f(z)| \leq |z|$ for all $z$ in particular by letting $z = 0$, we see that $f(z) = 0$. Therefore, $a = 0$. Also, since we assume that $f'(0) = 1$, we have that $b = 1$. Consequently, $f(z) = z$ for all $z \in \mathbb{C}$.

**Solution 2.** This solution is very similar to Solution 1 and we merely sketch the details. Apply Cauchy’s estimate to $f$ as in Solution 1, but with $n = 1$ to obtain that

$$|f'(z_0)| \leq \frac{(|z_0| + R)}{R}.$$ 

By letting $R \to \infty$ in this inequality we deduce that

$$|f'(z_0)| \leq 1.$$ 

Since this last inequality holds for all $z_0 \in \mathbb{C}$, it follows $f'$ is bounded. But since $f$ is entire, so also $f'$ is entire (cf. Theorem 1 on pg. 168). Therefore, as $f'$ is a bounded entire function, it follows by Liouville’s Theorem that $f'$ is constant, say $f'(z) = b$ for all $z \in \mathbb{C}$. Since antiderivatives are unique up to a constant, it follows that $f$ has the form $f(z) = a + bz$ just as in Solution 1.