1. Compute:

(i) \((1 - i)(2 - i)\)

(ii) \(\frac{1}{3 - 4i}\)

(iii) \((1 + i)^{15}\)

Solution.

(i)

\[
(1 - i)(2 - i) = 2 - 2i - i + i^2 \\
= 2 - 2i - i - 1 \\
= 1 - 3i.
\]

(ii)

\[
\frac{1}{3 - 4i} = \frac{1}{3 - 4i} \frac{3 + 4i}{3 + 4i} \\
= \frac{3 + 4i}{9 - (4i)^2} \\
= \frac{3 + 4i}{9 + 16} \\
= \frac{3}{25} + \frac{4}{25}i.
\]
(iii) $1 + i = \sqrt{2} e^{\frac{\pi i}{4}}$. Therefore,

$$(1 + i)^{15} = (\sqrt{2} e^{\frac{\pi i}{4}})^{15}$$

$$= (\sqrt{2})^{15} e^{\frac{15\pi i}{4}}$$

$$= 128\sqrt{2} \left( \cos\left(\frac{15\pi}{4}\right) + \sin\left(\frac{15\pi}{4}\right) i \right)$$

$$= 128\sqrt{2} \left( \cos\left(\frac{7\pi}{4}\right) + \sin\left(\frac{7\pi}{4}\right) i \right)$$

$$= 128\sqrt{2} \left( \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i \right)$$

$$= 128 - 128i.$$

Alternately,

$$(1 + i)^2 = 1 + 2i + i^2 = 1 + 2i - 1 = 2i,$$

so that

$$(1 + i)^4 = ((1 + i)^2)^2 = (2i)^2 = -4.$$

Consequently,

$$(1 + i)^{15} = ((1 + i)^4)^3(1 + i)^2(1 + i)$$

$$= (-4)^3 (2i) (1 + i)$$

$$= -64 (-2 + 2i)$$

$$= 128 - 128i.$$

2. Find all solutions to the equation $z^4 = -1$.

Solution. If we let $z = re^{i\theta}$ and note that $-1 = e^{\pi i}$, the equation becomes

$$r^4 e^{4\theta i} = e^{\pi i}.$$

Consequently, $z = re^{i\theta}$ satisfies $z^4 = -1$ if and only if $r = 1$ and

$$4\theta i = \pi i + 2k\pi i$$
for some \( k \in \mathbb{Z} \). Therefore, the solutions to \( z^4 = -1 \) have the form
\[
z_k = e^{\frac{2k+1}{4} \pi i}, \quad k = 0, \pm 1, \pm 2, \ldots .
\]
But since \( z_{k+4} = z_k \), there are only 4 different such complex numbers, \( z_0, z_1, z_2, \) and \( z_3 \), or in rectangular coordinates,
\[
\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i, \quad -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i, \quad -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i, \quad \text{and} \quad \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i.
\]

3. (i) Let \( S \) be a set of complex numbers. Give the definition for a point \( z_0 \in \mathbb{C} \) to be an interior point of \( S \).

(ii) Prove that if \( D = \{ z \in \mathbb{C} : |z| < 1 \} \), then \( D \) is open (i.e., every point in \( D \) is an interior point of \( D \)).

Solution.

(i) \( z_0 \) is an interior point of \( S \) if there exists a neighborhood of \( z_0 \) that lies in \( S \), i.e., there exists \( \epsilon > 0 \) such that \( z \in S \) whenever \( |z - z_0| < \epsilon \).

(ii) Fix \( z_0 \in D \). We need to show that there exists \( \epsilon > 0 \) such that \( z \in D \) whenever \( |z - z_0| < \epsilon \). Since \( z_0 \in D \), by the definition of \( D \), \( |z_0| < 1 \). Therefore, if we let \( \epsilon = 1 - |z_0|, \epsilon > 0 \). Furthermore, if \( |z - z_0| < \epsilon \), then
\[
|z| = |z_0 + (z - z_0)| \\
\leq |z_0| + |z - z_0| \\
< |z_0| + \epsilon \\
= |z_0| + (1 - |z_0|) \\
= 1.
\]

Therefore, \( z \in D \) whenever \( |z - z_0| < \epsilon \).

4. Let \( D \) be the domain consisting of all \( z = re^{i\theta} \) where \( r > 0 \) and \( 0 < \theta < 2\pi \) and define a complex valued function \( f \) on \( D \) by the formula
\[
f(z) = \sqrt{r} \ e^{i\theta}.
\]
Prove that $f$ is a branch of $\sqrt{z}$ on $D$, i.e., $f$ is analytic on $D$ and $f(z)^2 = z$ for all $z \in D$.

**Solution.** We have that $f = u + iv$ where

$$ u(r, \theta) = \sqrt{r} \cos(\frac{\theta}{2}) $$

and

$$ v(r, \theta) = \sqrt{r} \sin(\frac{\theta}{2}). $$

Observe that $u$ and $v$ are differentiable on $D$. Therefore, $f$ will analytic on $D$ if the Cauchy-Riemann equations hold on $D$ (cf. Theorem on pg. 69 of the text). To verify that $u$ and $v$ satisfy the Cauchy-Riemann equation hold, we work in polar coordinates (cf. Equation (6) pg. 69). We have that

$$ u_r = \frac{1}{2 \sqrt{r}} \cos(\frac{\theta}{2}) $$

$$ u_\theta = -\frac{\sqrt{r}}{2} \sin(\frac{\theta}{2}), $$

$$ v_r = \frac{1}{2 \sqrt{r}} \sin(\frac{\theta}{2}), $$

and

$$ v_\theta = \frac{\sqrt{r}}{2} \cos(\frac{\theta}{2}). $$

Therefore,

$$ ru_r = r \left( \frac{1}{2 \sqrt{r}} \cos(\frac{\theta}{2}) \right) = \frac{\sqrt{r}}{2} \cos(\frac{\theta}{2}) = v_\theta $$

and

$$ rv_r = r \left( \frac{1}{2 \sqrt{r}} \sin(\frac{\theta}{2}) \right) = \frac{\sqrt{r}}{2} \sin(\frac{\theta}{2}) = -u_\theta, $$

i.e., the Cauchy-Riemann equations hold. This establishes that $f$ is analytic on $D$. 
To see that $f(z)^2 = z$ for all $z \in D$, fix $z = re^{i\theta} \in D$. Then

$$f(z)^2 = \left(\sqrt{r} e^{i\frac{\theta}{2}}\right)^2 = \left(\sqrt{r}\right)^2 (e^{i\frac{\theta}{2}})^2 = re^{i\theta} = z.$$ 

5. Prove that if $f$ is entire, $f(0) = 1$, and $f'(z) = 2f(z)$ for all $z \in \mathbb{C}$, then $f(z) = e^{2z}$ for all $z \in \mathbb{C}$.

Solution. We employ the trick that was used in class to show that $f(z) = e^z$ is the unique entire function satisfying $f(0) = 1$ and $f'(z) = f(z)$ for all $z \in \mathbb{C}$.

Suppose $f$ is an entire function, $f(0) = 1$, and $f'(z) = 2f(z)$ for all $z \in \mathbb{C}$. Define a function $g$ on $\mathbb{C}$ by the formula

$$g(z) = \frac{f(z)}{e^{2z}}, \quad z \in \mathbb{C}.$$ 

Since $f(z)$ and $e^{2z}$ are both entire functions, and $e^{2z} \neq 0$ for all $z \in \mathbb{C}$, $g(z)$ is an entire function. Furthermore,

$$g'(z) = \frac{d}{dz} \frac{f(z)}{e^{2z}} = \frac{f'(z)e^{2z} - f(z)2e^{2z}}{(e^{2z})^2}$$

$$= \frac{2f(z) e^{2z} - f(z)2e^{2z}}{e^{4z}} = 0$$

for all $z \in \mathbb{C}$. Consequently, since $\mathbb{C}$ is a domain, it follows that $g$ is constant (cf. Theorem on page 73). But if $c \in \mathbb{C}$ and $g(z) = c$ for all $z \in \mathbb{C}$, then

$$f(z) = ce^{2z}$$

for all $z \in \mathbb{C}$. In particular, by letting $z = 0$, we find that

$$1 = f(0) = ce^0 = c.$$

Hence, $f(z) = e^{2z}$ for all $z \in \mathbb{C}$, as was to be proved.