Math 120 A Practice Midterm 2 Solutions

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1 Let \( f(z) \) denote the principle branch of \( z^i \).

(i) Give the definition of \( f(z) \).

(ii) Show that \( f(z) \) is analytic on its domain of definition and compute its derivative.

(iii) Compute \( f(1) \) and \( f(i) \).

Solution.

(i) \( f(z) = e^{i \log z} \) where \( z = re^{i\theta} \) satisfies \( r > 0 \) and \( -\pi < \theta < \pi \).

(ii) \( \log z \) is analytic on \( r > 0 \) and \( -\pi < \theta < \pi \), and \( e^z \) is entire. Therefore, as the composition of analytic functions is analytic, \( f(z) \) is analytic on \( r > 0 \) and \( -\pi < \theta < \pi \). By the Chain Rule,

\[
 f'(z) = \frac{d}{dz} e^{i \log z} = e^{i \log z} \frac{d}{dz} i \log z = i e^{i \log z} \frac{1}{z} = i e^{i \log z} \frac{z}{e^{i \log z}} = i e^{(i-1) \log z}.
\]

Thus, if both \( z^i \) and \( z^{i-1} \) are defined using the principle branch of \( \log z \), then

\[
 \frac{d}{dz} z^i = iz^{i-1}.
\]

(iii) Since \( \log 1 = 0 \) and \( \log i = \frac{\pi}{2} i \),

\[
 f(1) = e^{i \log 1} = e^0 = 1
\]

and

\[
 f(i) = e^{i \log i} = e^{i \frac{\pi}{2} i} = e^{-\frac{\pi}{2}}.
\]
2. Find all solutions to the equation \( \cosh z = -2 \).

**Solution.** By definition, \( \cosh z = \frac{1}{2}(e^z + e^{-z}) \). Therefore, we wish to solve

\[
\frac{e^z + e^{-z}}{2} = -2,
\]

or equivalently, after multiplying by \( e^z \) and regrouping,

\[
e^{2z} + 4e^z + 1 = 0.
\]

Therefore, by the quadratic formula,

\[
e^z = \frac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}.
\]

Since \(-2 - \sqrt{3} = (2 + \sqrt{3})(-1) \) and \(-2 + \sqrt{3} = (2 - \sqrt{3})(-1) \), we find that either

\[
z = \ln(2 + \sqrt{3}) + \pi i + 2n\pi i, \quad n = 0, \pm 1, \pm 2, \ldots
\]

or

\[
z = \ln(2 - \sqrt{3}) + \pi i + 2n\pi i, \quad n = 0, \pm 1, \pm 2, \ldots.
\]

3. Let \( C_1 \) be the triangle with vertices 0, 1 and \( i \), oriented positively, and let \( C_2 \) be the right half of the circle \( |z| = 2 \), oriented counterclockwise. Compute the following contour integrals directly from the definition.

(i) \[ \int_{C_1} z^2 \, dz \]

(ii) \[ \int_{C_2} \log z \, dz \]

**Solution.**

(i) For \( z_1, z_2 \) a pair of points in \( \mathbb{C} \), let \([z_1, z_2]\) denote the contour parametrized by \( z(t) = z_1 + t(z_2 - z_1), \) \( 0 \leq t \leq 1 \). It follows that if \( z_1 \) and \( z_2 \) are complex
numbers, then
\[
\int_{z_1,z_2} z^2 \, dz = \int_{0}^{1} (z_1 + t(z_2 - z_1))^2(z_2 - z_1) \, dt \\
= (z_2 - z_1) \int_{0}^{1} (z_1^2 + 2z_1(z_2 - z_1)t + (z_2 - z_1)^2t^2) \, dt \\
= (z_2 - z_1)(z_1^2 + z_1(z_2 - z_1) + 1/3(z_2 - z_1)^2) \\
= 1/3(z_2 - z_1)(z_1^2 + z_1(z_2 + z_1)) \\
= 1/3(z_2^3 - z_1^3)
\]

Remark: Note that this formula follows immediately from the Antiderivative Theorem.

Using this formula we have that
\[
\int_{C_1} z^2 \, dz = \int_{[0,1]} z^2 \, dz + \int_{[1,i]} z^2 \, dz + \int_{[i,0]} z^2 \, dz \\
= 1/3 (1^3 - 0^3) + 1/3 (i^3 - 1^3) + 1/3 (0^3 - i^3) \\
= 1/3 + (-1/3 i - 1/3) + 1/3 i \\
= 0.
\]

Remark: One could just as well directly calculate \( \int_{[0,1]} z^2 \, dz \), \( \int_{[1,i]} z^2 \, dz \), and \( \int_{[i,0]} z^2 \, dz \) without using the general formula for \( \int_{z_1,z_2} z^2 \, dz \) derived above.

Remark: Note that \( \int_{C_1} z^2 \, dz = 0 \) by Cauchy’s Theorem. However, this would not be a correct way to solve the problem as you are asked to compute the contour integral directly from the definition.
(ii) $C_2$ can be parametrized by $z(t) = 2e^{it}, \ -\pi/2 \leq t \leq \pi/2$. Consequently,

$$\int_{C_2} \log z \, dz = \int_{-\pi/2}^{\pi/2} \log (2e^{it}) 2ie^{it} \, dt$$

$$= \int_{-\pi/2}^{\pi/2} (\ln 2 + it) 2ie^{it} \, dt$$

$$= (2 \ln 2) i \int_{-\pi/2}^{\pi/2} e^{it} \, dt - 2 \int_{-\pi/2}^{\pi/2} te^{it} \, dt$$

$$= (2 \ln 2) i \left( -ie^{it} \right) \bigg|_{-\pi/2}^{\pi/2} - 2 \left( (-it + 1)e^{it} \right) \bigg|_{-\pi/2}^{\pi/2}$$

$$= (2 \ln 2) i \left( -i \cdot i - i \cdot (-i) \right) - 2 \left( (-i\pi/2 + 1)i - (-i(-\pi/2) + 1)(-i) \right)$$

$$= (2 \ln 2) i \left( 1 + 1 \right) - 2 \left( (\pi/2 + i) - (\pi/2 - i) \right)$$

$$= 4(\ln 2 - 1)i. \quad \text{Remark: this integral is good practice, but is probably too time consuming for an in class exam.}$$

4. Let $C_R$ denote the upper half of the circle $|z| = R$, parametrized in the counterclockwise direction. Show that if $t < -1$, then regardless of the choice of branch for $z^t$,

$$\lim_{R \to \infty} \int_{C_R} z^t \, dz = 0. \quad \text{Solution.} \quad z^t \text{ is defined by}$$

$$z^t = e^{t \log z}. \quad \text{Now,} \quad |e^z| = e^{\Re z} \text{ and } \Re(\log z) = \log |z|. \quad \text{Therefore, regardless of the branch of } \log z \text{ that is chosen in the definition of } z^t,$$

$$|z^t| = |e^{t \log z}| = e^{\Re(t \log z)} = e^{t \ln |z|} = |z|^t.$$

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Consequently, if \( z = Re^{i\theta} \in C_R \),

\[ |z|^t = |z|^t = R^t. \]

As a consequence (cf. Theorem on page 136 of the text),

\[ | \int_{C_R} z^t \, dz | \leq (R^t)(\pi R) = \pi R^{t+1}. \]

Since \( R^{t+1} \to 0 \) as \( R \to \infty \) when \( t < -1 \), it follows that \( \int_{C_R} z^t \, dz \to 0 \) as \( R \to \infty \) when \( t < -1 \).

5. Let \( C \) be the upper half of the ellipse \( 4x^2 + \pi^2y^2 = \pi^2 \), parametrized counterclockwise. Compute

\[ \int_C \cos z \, dz \]

two ways; using the Antiderivative Theorem and using Cauchy’s Theorem.

**Solution.** Note that \( \frac{d}{dz}(\sin z) = \cos z \) and the contour \( C \) has initial point \( \pi/2 \) and terminal point \( -\pi/2 \). Therefore by the Antiderivative Theorem,

\[ \int_C \cos z \, dz = (\sin(-\pi/2)) - (\sin \pi/2) = -1 - 1 = -2. \]

Let \([-\pi/2, \pi/2]\) denote the contour, \( z(t) = -\pi/2 + t, 0 \leq t \leq \pi \). Then

\[ C_1 = C + [-\pi/2, \pi/2] \]

is a closed contour and \( \cos z \) is analytic on and inside \( C_1 \) (in fact, \( \cos z \) is
entire). Therefore, by Cauchy’s Theorem, $\int_{C_1} \cos z \, dz = 0$. But

$$\int_{C_1} \cos z \, dz = \int_{C} \cos z \, dz + \int_{[-\pi/2, \pi/2]} \cos z \, dz$$

$$= \int_{C} \cos z \, dz + \int_{0}^{\pi} \cos(-\pi/2 + t) \, dt$$

$$= \int_{C} \cos z \, dz + \sin(-\pi/2 + t) \bigg|_{0}^{\pi}$$

$$= \int_{C} \cos z \, dz + 2.$$ 

Therefore, $\int_{C} \cos z \, dz = -2$. 

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