Math 142B Midterm Solutions

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1. State the Weierstrass Approximation Theorem.

Solution. If $f : [a, b] \rightarrow \mathbb{R}$ is continuous and $\epsilon > 0$, then there exists a polynomial $p$ such that

$$|f(x) - p(x)| < \epsilon \quad \text{for all } x \in [a, b].$$

2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function defined on $\mathbb{R}$ with the property that

$$\int_a^b f = b - a \quad (1)$$

whenever $a$ and $b$ are real numbers with $a < b$. Prove that $f(x) = 1$ for all $x \in \mathbb{R}$.

Solution 1. Fix $a \in \mathbb{R}$ and for $x \in \mathbb{R}$, let $F(x)$ be defined on $\mathbb{R}$ by the formula

$$F(x) = \int_a^x f.$$

By the Fundamental Theorem of Calculus (Version 2) it follows that $F$ is differentiable on $\mathbb{R}$ and $F'(x) = f(x)$ for all $x \in \mathbb{R}$. But (1) with $b = x$ implies that $F(x) = x - a$ for all $x \in \mathbb{R}$. Therefore,

$$f(x) = F'(x) = 1 \quad \text{for all } x \in \mathbb{R}.$$
Solution 2. Fix $a, b \in \mathbb{R}$ with $a < b$. By the Mean Value Theorem for Integrals there exists $x_0 \in [a, b]$ such that

$$f(x_0) = \frac{1}{b - a} \int_a^b f.$$ 

But (1) implies that

$$\frac{1}{b - a} \int_a^b f = \frac{1}{b - a} (b - a) = 1.$$ 

This proves that if $a$ and $b$ are arbitrary real numbers with $a < b$ then there exists $x_0 \in \mathbb{R}$ such that $a < x_0 < b$ and $f(x_0) = 1$, or stated differently, the set $S$ defined by

$$S = \{ x \in \mathbb{R} \mid f(x) = 1 \}$$

is a dense subset of $\mathbb{R}$. Therefore, as $f$ is continuous on $\mathbb{R}$, $f(x) = 1$ for all $x \in \mathbb{R}$.

3. Prove that

$$1 + \frac{x}{3} - \frac{x^2}{9} < (1 + x)^{1/3} < 1 + \frac{x}{3} \quad \text{if } x > 0.$$ 

Solution. Let $f(x) = (1 + x)^{1/3}$. Then

$$f'(x) = \frac{1}{3} (1 + x)^{-2/3},$$

$$f''(x) = -\frac{2}{9} (1 + x)^{-5/3},$$

$$f'''(x) = \frac{10}{27} (1 + x)^{-8/3}.$$ 

As $f(0) = 1$, $f'(0) = \frac{1}{3}$, and $f''(0) = -\frac{2}{9}$,

$$p_1(x) = 1 + \frac{x}{3} \quad \text{and} \quad p_2(x) = 1 + \frac{x}{3} - \frac{x^2}{9}.$$
By the Lagrange Remainder Theorem with \( n = 1 \) we have that,

\[
(1 + x)^{1/3} = p_1(x) + R_1(x) = 1 + \frac{x}{3} + \frac{f''(c)}{2!}x^2 = 1 + \frac{x}{3} - \frac{2}{9}(1 + c)^{-5/3}x^2
\]

where \( c \) lies between 0 and \( x \). Since \( x > 0 \) it follows that

\[
\frac{2}{9}(1 + c)^{-5/3}x^2 > 0,
\]

so that

\[
(1 + x)^{1/3} < 1 + \frac{x}{3}.
\]

Similarly, by the Lagrange Remainder Theorem with \( n = 2 \) we have that,

\[
(1 + x)^{1/3} = p_2(x) + R_2(x) = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{f'''(c)}{3!}x^3 = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{10}{27}(1 + c)^{-8/3}x^3
\]

where \( c \) lies between 0 and \( x \). Since \( x > 0 \) it follows that

\[
\frac{10}{27}(1 + c)^{-8/3}x^3 > 0,
\]

so that

\[
1 + \frac{x}{3} - \frac{x^2}{9} < (1 + x)^{1/3}.
\]

4. Show that the sequence \( \{c_n\} \) defined by

\[
c_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n + 1), \quad n \geq 1,
\]

is strictly monotonically increasing.
Solution. Observe that for $n \geq 2$

$$c_n - c_{n-1} = (1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n + 1)) - (1 + \frac{1}{2} + \ldots + \frac{1}{n-1} - \ln n)$$

$$= \frac{1}{n} - (\ln(n + 1) - \ln n).$$

Therefore, $\{c_n\}$ will be strictly monotonically increasing if

$$0 < \frac{1}{n} - (\ln(n + 1) - \ln n) \quad \text{for all } n \geq 2. \quad (2)$$

We give 4 proofs of this inequality.

Proof 1. We use the inequality

$$0 < x - \ln x$$

which is valid for all $x > 0$ (cf. Quiz 3). Letting $x = 1/n$ in this inequality yields that if $n \geq 2$, then

$$\frac{1}{n} - (\ln(n + 1) - \ln n) = \frac{1}{n} - \ln \frac{n + 1}{n}$$

$$= \frac{1}{n} - \ln(1 + \frac{1}{n})$$

$$> 0,$$

i.e., (2) holds.

Proof 2.

$$\ln(n + 1) - \ln n = \int_{n}^{n+1} \frac{1}{t} dt$$

is the area under the graph of $1/t$ lying over the interval $[n, n + 1]$ and

$$\frac{1}{n} = \max_{t \in [n, n+1]} \frac{1}{t}$$

is the maximum of $1/t$ on the interval $[n, n + 1]$. Since clearly the area under the graph of $1/t$ is less that the maximum of $1/t$, (2) holds.
Proof 3. By the Mean Value Theorem for Derivatives applied to the function \( \ln x \) on the interval \([n, n + 1]\), there exists an \( c \in (n, n + 1) \) such that

\[
\ln(n + 1) - \ln n = \frac{\ln(n + 1) - \ln n}{(n + 1) - n} = \frac{1}{c}.
\]

Since \( c \in (n, n + 1) \),

\[
\frac{1}{c} < \frac{1}{n},
\]

and therefore, (2) holds.

Proof 4. By the Mean Value Theorem for Integrals applied to the function \( 1/x \) on the interval \([n, n + 1]\), there exists \( x_0 \in (n, n + 1) \) such that

\[
\ln(n + 1) - \ln n = \frac{1}{(n + 1) - n} \int_n^{n+1} \frac{1}{x} \, dx = \frac{1}{x_0}.
\]

Since \( x_0 \in (n, n + 1) \),

\[
\frac{1}{x_0} < \frac{1}{n},
\]

and again we see that (2) holds.

5. Suppose that \( f : [a, b] \to \mathbb{R} \) is integrable. Use the Archimedes-Riemann Theorem to show that \( 2f \) is integrable and \( \int_a^b 2f = 2 \int_a^b f \).

Solution. Since \( f \) is assumed to be integrable on \([a, b]\), the Archimedes-Riemann Theorem implies that there exists an Archimedean sequence of partitions for \( f \) on \([a, b]\), i.e., a sequence \( \{P_n\} \) of partitions of \([a, b]\) such that

\[
\lim_{n \to \infty} \left( U(f, P_n) - L(f, P_n) \right) = 0. \quad (3)
\]

and

\[
\lim_{n \to \infty} U(f, P_n) = \int_a^b f. \quad (4)
\]
Noting that
\[ U(2f, P_n) = 2U(f, P_n) \quad \text{and} \quad L(2f, P_n) = 2L(f, P_n), \]
it follows using (3) that
\[ \lim_{n \to \infty} (U(2f, P_n) - L(2f, P_n)) = \lim_{n \to \infty} 2(U(f, P_n) - L(f, P_n)) = 0, \]
i.e., \( \{P_n\} \) is an Archimedean sequence of partitions for \( 2f \) on the interval \([a, b]\). Therefore, by the Archimedes-Riemann Theorem, \( 2f \) is integrable on \([a, b]\) and
\[ \int_a^b 2f = \lim_{n \to \infty} U(2f, P_n) = \lim_{n \to \infty} 2U(f, P_n) \overset{(4)}{=} 2 \int_a^b f. \]