Math 142B Quiz 1 Solution

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Part 1. Prove that if \( f \) and \( g \) are bounded real valued functions defined on a set \( S \subseteq \mathbb{R} \) and \( f(x) \leq g(x) \) for all \( x \in S \), then
\[
\inf_{x \in S} f(x) \leq \inf_{x \in S} g(x).
\] (1)

Solution. Let \( m = \inf_{x \in S} f(x) \) so that \( m \) is the greatest lower bound of the set \( R_f = \{ f(x) \mid a \leq x \leq b \} \). Since in particular, \( m \) is a lower bound for \( R_f \), if \( x \in [a, b] \), then
\[
m \leq f(x) \leq g(x).
\]
This proves that \( m \) is a lower bound for \( R_g = \{ g(x) \mid a \leq x \leq b \} \). Therefore, since \( \inf_{x \in S} g(x) \) is the greatest lower bound of \( R_g \), \( m \leq \inf_{x \in S} g(x) \), i.e., (1) holds.

Part 2. Use Part 1 to prove that if \( f \) and \( g \) are integrable functions on \([a, b] \) satisfying \( f(x) \leq g(x) \) for all \( x \in [a, b] \), then
\[
\int_a^b f \leq \int_a^b g.
\]

Solution. Choose a sequence of partitions \( \{P_n\} \) of \([a, b] \) that is an Archimedean sequence for both \( f \) and \( g \). Using Part 1 we have that
\[
L(f, P_n) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) \leq \sum_{i=1}^{n} m_i(g)(x_i - x_{i-1}) = L(g, P_n)
\]
But \( L(f, P_n) \to \int_a^b f \) and \( L(g, P_n) \to \int_a^b g \). Therefore, \( \int_a^b f \leq \int_a^b g \).