Suppose that a continuous function \( f : [a, b] \to \mathbb{R} \) has the property that \( \int_c^d f \leq 0 \) whenever \( a \leq c < d \leq b \). Prove that \( f(x) \leq 0 \) for all \( x \in [a, b] \) in the following two ways.

(i) Using the Basic Fact about Continuous Functions from the Practice Quiz.

(ii) Using the Mean Value Theorem.

**Proof using the Basic Fact.** We argue by contradiction. Accordingly, assume that there exists a point \( x_0 \in [a, b] \) such that \( f(x_0) > 0 \). By the basic fact there exist \( \rho > 0 \) and \( c, d \in [a, b] \) with \( c < d \) and \( f(x) \geq \rho \) for all \( x \in [c, d] \). But then

\[
\int_c^d f \geq \int_c^d \rho = \rho(d - c) > 0,
\]

contradicting the assumption that \( \int_c^d f \leq 0 \) whenever \( a \leq c < d \leq b \).

**Proof using the Mean Value Theorem.** Suppose that a continuous function \( f : [a, b] \to \mathbb{R} \) has the property that \( \int_c^d f \leq 0 \) whenever \( a \leq c < d \leq b \). Fix a point \( x \in [a, b] \). Construct a pair of sequences \( \{c_n\} \) and \( \{d_n\} \) with the following properties

\[
\begin{align*}
& a \leq c_n < d_n \leq b \quad \text{for all } n \\
& x \in [c_n, d_n] \quad \text{for all } n \\
& \lim_{n \to \infty} (d_n - c_n) = 0
\end{align*}
\]

By the assumption that \( \int_c^d f \leq 0 \) whenever \( a \leq c < d \leq b \), we have from (??) that

\[
\int_{c_n}^{d_n} f \leq 0 \quad \text{for all } n.
\]

For each fixed \( n \) apply the Mean Value Theorem to the integral in (4) to obtain a point \( x_n \in [c_n, d_n] \) such that

\[
f(x_n) = \frac{1}{d_n - c_n} \int_{c_n}^{d_n} f \leq 0.
\]

Since \( x_n \in [c_n, d_n] \) for all \( n \) it follows from (2) and (3) that \( x_n \to x \). Therefore, since \( f \) is assumed to be continuous, we have that \( f(x_n) \to f(x) \) as \( n \to \infty \). Since \( f(x_n) \leq 0 \) for each \( n \), it follows that \( f(x) \leq 0 \).