1. Let \( \{p_n\} \) be a sequence of polynomials such that \( p_n \) converges uniformly to a function \( \phi \) on \( C = \{ \lambda \in \mathbb{C} \mid |\lambda| = 1 \} \). Prove that there exists \( f \in H(\mathbb{D}) \) such that \( p_n \to f \) in \( H(\mathbb{D}) \).

Solution. Since \( p_n \to \phi \) uniformly on \( C \), the sequence \( \{p_n\} \) is Cauchy in \( C(C, \mathbb{C}) \), the space of continuous functions on \( C \).

We claim that \( \{p_n\} \) is also Cauchy in \( H(\mathbb{D}) \). Fix \( \epsilon > 0 \) and a compact subset \( K \) of \( \mathbb{D} \). Choose \( N \) such that if \( m,n > N \),

\[
\max_{\lambda \in C} |p_n(\lambda) - p_m(\lambda)| < \epsilon.
\]

By the maximum principle, it follows that if \( m,n > N \), then

\[
\max_{z \in K} |p_n(z) - p_m(z)| < \epsilon.
\]

This proves that \( \{p_n\} \) is Cauchy in \( H(\mathbb{D}) \).

Since \( H(\mathbb{D}) \) is complete, there exists an \( f \in H(\mathbb{D}) \) such that \( p_n \to f \) in \( H(\mathbb{D}) \).

2. Let \( G \) be a simply connected region in the plane and assume that \( f \) is a conformal map from \( G \) to \( \mathbb{D} \) (i.e., \( f : G \to \mathbb{D} \) is an analytic bijection). Prove that if \( g \) is any other conformal map from \( G \) to \( \mathbb{D} \) then there exist \( c, \alpha \in \mathbb{C} \) with \( |c| = 1 \) and \( \alpha \in \mathbb{D} \) such that \( g(z) = c\phi_\alpha(f(z)) \).

Solution. A basic fact in the text (Theorem 2.5 pg. 132) was that \( h : \mathbb{D} \to \mathbb{D} \) is a conformal map from the disc to the disc if and only if there exist \( c, \alpha \in \mathbb{C} \) with \( |c| = 1 \) and \( \alpha \in \mathbb{D} \) such that \( h(w) = c\phi_\alpha(w) \) for all \( w \in \mathbb{D} \). Here, \( \phi_\alpha \) is defined by

\[
\phi_\alpha(w) = \frac{w - \alpha}{1 - \bar{\alpha}w}.
\]

As a consequence, if \( f \) and \( g \) are two conformal maps from \( G \) to \( \mathbb{D} \), then, as \( g \circ f^{-1} \) is a conformal map from \( \mathbb{D} \) to \( \mathbb{D} \), there exist \( c, \alpha \in \mathbb{C} \) with \( |c| = 1 \) and \( \alpha \in \mathbb{D} \) such that \( g \circ f^{-1}(w) = c\phi_\alpha(w) \) for all \( w \in \mathbb{D} \). The desired result follows by letting \( w = f(z) \).

3. Suppose that \( f \) is analytic on \( \mathbb{D} \) with \( |f(z)| \leq 1 \) for all \( z \in \mathbb{D} \). If \( f = 0 \) at the distinct
points $a_1, \ldots, a_n \in \mathbb{D}$, prove the inequality,

$$|f(z)| \leq \prod_{j=1}^{n} \left| \frac{z - a_j}{1 - \overline{a_j}z} \right|,$$

for all $z \in \mathbb{D}$. If $f$ has a double 0 at $a_j$ for some $j$, prove that the inequality is strict for all $z \in \mathbb{D}$.

**Solution.** Let

$$\Pi(z) = \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j}z},$$

and define a function $g$ by the formula,

$$g(z) = \frac{f(z)}{\Pi(z)}.$$

The denominator in the expression defining $g$ has simple zeros at each $a_j$, but, as $f(a_j) = 0$ for $j = 1, \ldots, n$, these singularities of $g$ are removable.

Noting that, if $\phi_\alpha$ is defined by (1), then $\phi_\alpha$ has the properties

$$\phi_\alpha$$

is analytic in a neighborhood of $\mathbb{D}^-$, and

$$|\phi_\alpha(w)| = 1$$

whenever $|w| = 1$,

it follows that

$$\Pi = \prod_{j=1}^{n} \phi_{a_j}$$

is analytic on a neighborhood of $\mathbb{D}^-$ and $|\Pi(z)| = 1$ whenever $|z| = 1$. In particular, we have that

$$\lim_{r \to 1^{-}} |\Pi(z)| = 1.$$

Hence, by the maximum modulus principle,

$$\sup_{\mathbb{D}} |g(z)| = \lim_{r \to 1^{-}} \sup_{|z| = r} \left| \frac{f(z)}{\Pi(z)} \right| \leq \lim_{r \to 1^{-}} \sup_{|z| = r} \left| \frac{f(z)}{\Pi(z)} \right| = \lim_{r \to 1^{-}} \sup_{|z| = r} |f(z)| \leq 1.$$

Equivalently, if $z \in \mathbb{D}$,

$$|f(z)| \leq \prod_{j=1}^{n} \left| \frac{z - a_j}{1 - \overline{a_j}z} \right|,$$

as was to be shown.

If $f$ has a double zero at some $a_j$, $g$ is nonconstant, and the maximum principle implies that $|g(z)| < 1$ for all $z \in \mathbb{D}$. Thus,

$$|f(z)| < \prod_{j=1}^{n} \left| \frac{z - a_j}{1 - \overline{a_j}z} \right|,$$

for all $z \in \mathbb{D}$.
4. Let $G$ be an open set in $\mathbb{C}$ and let $F \subseteq H(G)$. Prove that if $F$ is locally bounded, then $F$ is locally Lipschitz, i.e., for each $a \in G$ there exist $r > 0$ and a constant $M$ such that $B(a;r) \subseteq G$ and $|f(z) - f(w)| \leq M|z - w|$ for all $z, w \in B(a;r)$.

**Solution.** Fix $a \in G$. Since $F$ is assumed locally bounded, there exists $r > 0$ and a constant $M$ such that $B(0;r) - \subseteq G$ and for all $f \in F$.

We claim that $F$ is uniformly Lipschitz on $B(a;r/2)$. To see this fix $z, w \in B(a;r/2)$ and let $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$. Then by the Cauchy Integral Formula, for each $f \in F$,

$$|f(z) - f(w)| = \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(\lambda)}{\lambda - z} \, d\lambda - \frac{1}{2\pi i} \int_{\gamma} \frac{f(\lambda)}{\lambda - w} \, d\lambda \right|$$

$$= \frac{1}{2\pi} \left| \int_{\gamma} \frac{(z - w)f(\lambda)}{(\lambda - z)(\lambda - w)} \, d\lambda \right|$$

$$\leq \frac{|z - w|}{2\pi} \max_{\lambda \in \gamma} \left| \frac{f(\lambda)}{(\lambda - z)(\lambda - w)} \right| \|\gamma\|$$

$$= \frac{|z - w|}{2\pi} \frac{M}{r(r/2)(r/2)} \frac{2\pi r}{r}$$

$$= \frac{4M}{r} |z - w|.$$

This proves that $F$ is uniformly Lipschitz on $B(a;r/2)$ - as was claimed.

5. Let $F$ be the collection of analytic functions on $\mathbb{D}$ whose power series expansion, $\sum_{n=0}^{\infty} a_n z^n$, satisfies $|a_n| \leq n$ for all $n \geq 0$. Prove that $F$ is a normal family.

**Solution.** By Montel’s theorem it is sufficient to show that $F$ is locally bounded. Fix $K$ compact in $\mathbb{D}$. Define $r = \sup_K |z| < 1$. Note that, if $f \in F$ and $z \in K$, then

$$|f(z)| = \left| \sum_{n=0}^{\infty} a_n z^n \right| \leq \sum_{n=0}^{\infty} |a_n| |z|^n \leq \sum_{n=0}^{\infty} nr^n = \frac{r}{(1 - r)^2}$$

Thus, $F$ is locally bounded.

6. Evaluate

$$\prod_{n=2}^{\infty} (1 - \frac{1}{n^2})$$

in the following two ways: (i) directly, and (ii) from the Weierstrass factorization of the sin function.

**Solution.**
(i) Note that
\[
\left(1 - \frac{1}{n^2}\right) = \left(\frac{n^2 - 1}{n^2}\right) = \left(\frac{(n - 1)(n + 1)}{n^2}\right).
\]
Hence,
\[
\prod_{n=2}^{N} \left(1 - \frac{1}{n^2}\right) = \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{2 \cdot 4}{3 \cdot 3} \cdot \frac{3 \cdot 6}{4 \cdot 4} \cdots \frac{(N - 1) \cdot (N + 1)}{N \cdot N} = \frac{N + 1}{2N}.
\]
Therefore,
\[
\lim_{n \to \infty} \prod_{n=2}^{N} \left(1 - \frac{1}{n^2}\right) = \lim_{N \to \infty} \frac{N + 1}{2N} = \frac{1}{2}.
\]

(ii) Using the factorization of the sine function,
\[
\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right), \quad z \in \mathbb{C},
\]
we have that
\[
\frac{\sin(\pi z)}{\pi z(1 - z^2)} = \prod_{n=2}^{\infty} \left(1 - \frac{z^2}{n^2}\right), \quad z \in \mathbb{C} \setminus \{-1, 0, 1\}.
\]
The left hand side of this equation has a singularity at 1, but it is removable. Also, the right hand side is entire. Thus,
\[
\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \lim_{z \to 1} \prod_{n=2}^{\infty} \left(1 - \frac{z^2}{n^2}\right)
\]
\[
= \lim_{z \to 1} \frac{\sin(\pi z)}{\pi z(1 - z^2)}
\]
\[
= \lim_{z \to 1} \frac{-1}{\pi z(1 + z)} \frac{\sin(\pi z)}{z - 1}
\]
\[
= -\frac{1}{2\pi} \lim_{z \to 1} \frac{\sin(\pi z)}{z - 1}
\]
\[
= -\frac{1}{2\pi} \left(\sin(\pi z)\right)'(1)
\]
\[
= -\frac{1}{2\pi} (-\pi)
\]
\[
= \frac{1}{2}.
\]

7. Let $G$ be a connected open set and let $\{f_n\}$ be a sequence in $H(G)$. Assume that $\prod_{n=1}^{\infty} f_n$ converges in $H(G)$ to a function $f$ which is not identically 0. Show that for $a \in G$, $f(a) = 0$ if and only if there exists an $n$ such that $f_n(a) = 0$. 
Solution. First suppose that \( f_n(a) = 0 \) for some \( n \). Then if \( N \geq n \), \( \prod_{m=1}^{N} f_m(a) = 0 \). Thus,

\[
f(a) = \prod_{n=1}^{\infty} f_n(a) = \lim_{N \to \infty} \prod_{n=1}^{N} f_n(a) = 0.
\]

Now suppose \( f(a) = 0 \). Since \( f \) is not identically 0, the zeros of \( f \) are isolated. Choose \( r > 0 \) such that \( f \) has no zeros in \( B(a; r) \setminus \{ a \} \). By Hurwitz’s Theorem, there is an \( N \) such that \( \prod_{N} f_n \) has a zero at some point \( w \in B(a; r) \). Thus, for some \( n \leq N \), \( f_n \) has a zero at \( w \). But then, by the result in the previous paragraph, \( f(w) = 0 \). Since \( f \) has no zeros in \( B(a; r) \setminus \{ a \} \), \( w = a \) and \( f_n(a) = 0 \).

8. Prove that for each \( \epsilon > 0 \), \( \frac{1}{z+i} + \sin z \) has infinitely many zeros in the region \( \{ z = x + iy \mid x > 0, |y| < \epsilon \} \).

Solution 1. Let \( \epsilon > 0 \) where without loss of generality we assume that \( \sinh \epsilon \leq 1 \). Observe that for \( z = x + iy \),

\[
\sin z = \frac{1}{2i} (e^{iz} - e^{-iz}) = \frac{1}{2i} (e^{ix-y} - e^{-ix+y}) = \frac{1}{2i} ((\cos x + i \sin x)e^{-y} - (\cos x - i \sin x)e^{y}) = \frac{1}{2i} (\cos x (e^{-y} - e^{y}) + i \sin x (e^{-y} + e^{y}) = \sin x \cosh y + i \cos x \sinh y.
\]

Hence,

\[
|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y. \tag{2}
\]

For \( m, n \) positive integers with \( m < n \) define a closed contour \( C_{m,n} \) by

\[
C_{m,n} = [2m\pi + \frac{\pi}{2} + \epsilon i, 2m\pi + \frac{\pi}{2} - \epsilon i] + [2m\pi + \frac{\pi}{2} - \epsilon i, 2m\pi + \frac{\pi}{2} + \epsilon i] + [2n\pi + \frac{\pi}{2} + \epsilon i, 2m\pi + \frac{\pi}{2} + \epsilon i] + [2n\pi + \frac{\pi}{2} + \epsilon i, 2m\pi + \frac{\pi}{2} + \epsilon i]
\]

Noting that \( \cosh^2 y \geq 1 \) for all \( y \) and \( \sin x = 1 \) on the vertical sides of \( C_{m,n} \), we see using (2) that

\[
|\sin z| \geq 1 \geq \sinh \epsilon
\]
on the vertical sides of \( C_{m,n} \). On the other hand on the horizontal sides of \( C_{m,n} \), (2) implies that

\[
|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \geq \sin^2 x + \cos^2 x \sinh^2 \epsilon
\]

5
\[
\geq \sin^2 x \sinh^2 \epsilon + \cos^2 x \sinh^2 \epsilon \\
= \sinh^2 \epsilon.
\]

Therefore, for all choices of positive integers \( m < n \),
\[
|\sin z| \geq \sinh \epsilon \quad \text{for all } z \in C_{m,n}. \tag{3}
\]

Now, since
\[
\max_{z \in C_{m,n}} \left| \frac{1}{z+i} \right| = \frac{1}{m+1} \to 0 \quad \text{as } m \to \infty,
\]
it follows that we may choose \( m \) such that
\[
\max_{z \in C_{m,n}} \left| \frac{1}{z+i} \right| < \sinh \epsilon \quad \text{for all } n > m.
\]

For this choice of \( m \), (3) implies that if \( n > m \)
\[
|\left( \frac{1}{z+i} + \sin z \right) - \sin z| = \left| \frac{1}{z+1} \right| < \sinh \epsilon \leq |\sin z|
\]
for all \( z \in C_{m,n} \). Hence, by Rouche’s Theorem, \( \frac{1}{z+i} + \sin z \) and \( \sin z \) have the same number of 0’s inside \( C_{m,n} \). Since \( \sin z \) has \( n - m \) 0’s inside \( C_{m,n} \) and \( n > m \) is arbitrary it follows that \( \frac{1}{z+i} + \sin z \) has infinitely many zeros in the region \( \{ z = x + iy \mid x > 0, |y| < \epsilon \} \).

**Solution 2.** Fix \( \epsilon > 0 \) and for \( n \geq 0 \), define
\[
G_n = \{ z = x + iy \mid 2\pi n < x < 2\pi(n+1), |y| < \epsilon \}, \quad f_n(z) = \frac{1}{z+2\pi n + i} + \sin z, \quad z \in \mathbb{C} \setminus \{-2\pi n - i\}.
\]
Noting that \( \sin(z + 2\pi n) = \sin z, \quad z \in \mathbb{C} \), it follows that
\[
f_n(z) = f_0(z + 2\pi n), \quad z \in G_0.
\]

Therefore,
\[
f_0 \text{ has a 0 in } G_n \iff f_n \text{ has a 0 in } G_0. \tag{4}
\]

Now, clearly, as \( (z + 2\pi n)^{-1} \to 0 \) uniformly on \( G_0 \), \( f_n \to \sin z \) uniformly on \( z \). Therefore, as \( \sin z \) has a 0 at \( \pi \in G_0 \), Hurwicz’s Theorem implies that there exists \( N \) such that
\[
n \geq N \implies f_n \text{ has a 0 in } G_0.
\]

Thus, using (4),
\[
n \geq N \implies f_0 \text{ has a 0 in } G_n.
\]

Since the sets \( G_n \) are pairwise disjoint subsets of \( G = \{ z = x + iy \mid x > 0, |y| < \epsilon \} \), this implies that \( f_0 \) has infinitely many 0’s in \( G \), as was to be proved.

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1 Curtesy of Jiaxi Nie.
Solution 3.2 Fix $\epsilon > 0$ and define functions $f$ and $g$ by
\[ f(z) = \sin z + \frac{1}{z + i} \quad \text{and} \quad g(z) = \sin z. \]
Choose $\delta$ satisfying $0 < \delta < \min\{\epsilon, \pi/2\}$ and for each $n \geq 1$ define a contour $\gamma_n$ by
\[ \gamma_n(t) = n\pi + \delta e^{it}, \quad 0 \leq t \leq 2\pi. \]
If we let
\[ \mu = \min_{w \in \gamma_1} |\sin w|, \]
then $\mu > 0$. Also, as $g(z + n\pi) = (-1)^n g(z)$,
\[ \forall_n \min_{w \in \gamma_n} |g(w)| = \mu. \]
On the other hand, if we let
\[ \epsilon_n = \max_{w \in \gamma_n} |f(w) - g(w)|, \]
then, $\epsilon_n \to 0$. Therefore, as $\mu > 0$, there exists $N$ such that
\[ n \geq N \implies \epsilon_n < \mu. \quad (5) \]
We claim that $f$ has exactly one 0 inside $\gamma_n$ for every $n \geq N$. To prove this claim fix $n \geq N$ and observe that if $z \in \gamma_n$, then (5) implies that
\[ |f(z) - g(z)| \leq \max_{w \in \gamma_n} |f(w) - g(w)| = \epsilon_n < \mu = \min_{w \in \gamma_n} |g(w)| \leq |g(z)|. \]
Therefore, since $g$ has exactly one 0 inside $\gamma_n$, Rouche’s Theorem implies that $f$ has exactly one 0 inside $\gamma_n$.

Finally, as $f$ has a 0 inside $\gamma_n$ for every $n \geq N$, the contours $\gamma_n$ are pairwise disjoint and lie in $\{z = x + iy \mid x > 0, |y| < \epsilon\}$, it follows that $f$ has infinitely many 0’s in $\{z = x + iy \mid x > 0, |y| < \epsilon\}$.

\[ \text{Curtesy of Sung Min Lee (John).} \]