Instructions: Do all 5 problems in Part I. Solve 3 problems from Part II. Be sure to indicate the problem from Part II you do not want graded.

Part I.

1. Show that the equation $e^z = 2z + 1$ has exactly one solution in $\mathbb{D}$.

2. Let $G$ be a connected open set in $\mathbb{C}$, $f$ a nonconstant analytic function on $G$, and $a \in G$. For $0 \leq r < \text{dist}(a, \mathbb{C} \setminus G)$, define $M(r)$ by

   $$M(r) = \max_{0 \leq \theta < 2\pi} |f(a + re^{i\theta})|.$$

   Prove that $M$ is a strictly increasing function. Is this result still true if $G$ is open but not connected?

3. Prove that if $f : \mathbb{D} \to \mathbb{D}$ is analytic, and $f(z)$ is not identically equal to $z$, then $f$ can have at most one fixed point in $\mathbb{D}$.

4. Let $\mathcal{F}$ consist of the analytic functions on $\mathbb{D}$ satisfying $f(1/3) = 0$ and $|f(z)| < 1$ for all $z \in \mathbb{D}$. Show that $\mathcal{F}$ is a compact subset of $H(\mathbb{D})$ and give a brief explanation how this implies that the two suprema,

   $$M_1 = \sup_{f \in \mathcal{F}} |f(\frac{2}{3})| \text{ and } M_2 = \sup_{f \in \mathcal{F}} |f'(\frac{1}{3})|,$$

   are attained.

5. Show that the series that defines the Riemann $\zeta$ function converges uniformly on compact subsets of $\text{Re} \ z > 1$. 
Part II.

1. Let $A$ denote the collection of continuous complex valued functions on $\mathbb{D}^-$ that are analytic on $\mathbb{D}$. Without using Runge’s Theorem, prove that $f \in A$ if and only if there exists a sequence of polynomials $p_n$ such that $p_n$ converges uniformly to $f$ on $\mathbb{D}^-$. 

2. Compute the $M_1$ and $M_2$ that are defined in problem 4 from Part I above.

3. Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function satisfying $f(0) = 0$. Prove that

$$|f(z) + f(-z)| \leq 2|z|^2$$

for all $z \in \mathbb{D}$. Further, show that this inequality is strict for all $z \in \mathbb{D} \setminus \{0\}$ unless $f(z) + f(-z) = 2cz^2$ for some $c \in \mathbb{C}$ with $|c| = 1$.

4. Let $G$ be an open set in $\mathbb{C}$. Show that there exists $f \in H(G)$ such that for each $a \in G$, the power series representation for $f$ at $a$ has radius of convergence $R = \text{dist}(a, \partial G)$.