Problem 1. True or False: For each statement below, determine whether it is true or false, and circle the appropriate letter. You do not need to justify your answer.

(**T** **F**) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$, then it is an orthogonal basis for W as well.

(**T** **F**) If A is a symmetric matrix and $A = P^{-1}DP$, where D is diagonal and P is invertible, then the columns of P are orthogonal.

(**T** **F**) If 0 is an eigenvalue of A then the columns of A are linearly dependent.

(**T** **F**) V and W are vector spaces, $T : V \to W$ is a linear transformation, and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is linearly independent in W.

(**T** **F**) Let A be an $(m \times n)$ matrix and $\mathbf{b} \in \mathbb{R}^m$. Let $\hat{\mathbf{b}}$ be the projection of $\mathbf{b}$ onto $\text{Col}(A)$. If $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$, then $\|A\hat{\mathbf{x}} - \mathbf{b}\| < \|A\mathbf{x} - \mathbf{b}\|$ for every $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} \neq \hat{\mathbf{x}}$.

(**T** **F**) If a system of linear equations has 2 solutions, then it has infinitely many solutions.

(**T** **F**) Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$ and $B = [\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3]$. If A is row equivalent to B, and $3\mathbf{a}_2 - \mathbf{a}_1 = \mathbf{a}_3$, then $3\mathbf{b}_2 - \mathbf{b}_1 = \mathbf{b}_3$.

(**T** **F**) If A is a square matrix, then $A^TA = AA^T$.

Problem 2. Let $U = \begin{bmatrix} .5 & -.5 \\ .5 & .5 \\ -.5 & .5 \\ -.5 & -.5 \end{bmatrix}$.

a) Show that U has orthonormal columns.

b) Find $\text{proj}_{\text{col}U}\mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 6 \end{bmatrix}$.

c) Find the distance from $\mathbf{b}$ to Col U.

Problem 3. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -4 \\ -2 \\ 0 \end{bmatrix}.$

a) Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent?

b) Find $\dim \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Problem 4. Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

a) Find all of the eigenvalues of A.

b) Find a basis for the eigenspace corresponding to each of the eigenvalues you found in part (a).

c) Is A diagonalizable? Why or why not?

Problem 5. Let $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$, and $D = \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix}$.

a) Find real numbers x_1, x_2, and x_3 such that $x_1 A + x_2 B + x_3 C = D$.

Hint: Use coordinates, and remember that $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ *is a basis for* $M_{2\times 2}$.

b) Is the solution you found unique? Why or why not?

Problem 6. a) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\ T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \text{and } T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

a) Write down the standard matrix representation for T.

b) Is T one-to-one? Why or why not?

c) Is T onto? Why or why not?

Problem 7. a) Give an example of a 2×3 matrix A such that none of the entries of A are zero, the columns of A are linearly dependent, and the columns of A do not span $\mathbb{R}^2$.

b) Give an example of a 3×2 matrix A such that the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one but not onto, and the columns of A are *not* orthogonal.

c) Give an example of a 3×3 matrix A such that $A = -A^T$ (these are called anti-symmetric matrices; they're really cool).

Problem 8. Let $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

a) Find a matrix B such that $B\mathbf{x} = \mathbf{e}_1$ and $B\mathbf{y} = \mathbf{e}_2$.

b) Is the matrix $[\mathbf{x} \;\; \mathbf{y}]$ invertible? Why or why not?