Problem 1. True or False: For each statement below, determine whether it is true or false, and circle the appropriate letter. You do not need to justify your answer. (5 points each)

(**T** **F**) If A and C are matrices (of the appropriate sizes) and $AC = 0$ then $A = 0$ or $C = 0$.
Solution: False. In fact this is only true if either A or C is invertible. Let $A = [0\ 1]$ and $C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for a counter example.

(**T** **F**) If $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4]$ is a 3×4 matrix and the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is linearly independent, then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^3$
Solution: True. If there are three linearly independent columns, then there must be three pivot positions in A, meaning that every row has a pivot position. Thus we always have a solution.

(**T** **F**) If A and B are $(n \times n)$ matrices such that A has a column of all zeros, then AB has a column of all zeros.
Solution: False. The actual statement is that if B has a column of zeros, then so does AB. For a counter example, let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(**T** **F**) If A is an $m \times n$ matrix and there exist vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^n$ such that $A\mathbf{x}_1 = A\mathbf{x}_2$, then for any $\mathbf{b}$ such that the equation $A\mathbf{x} = \mathbf{b}$ has a solution, the solution is not unique.
Solution: True. Note that if $A\mathbf{x}_1 = A\mathbf{x}_2$, then $A(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}$, so the homogeneous system has a nontrivial solution. Thus by the theorem in section 1.5, if $\mathbf{y}$ is a solution to $A\mathbf{x} = \mathbf{b}$, then so is $\mathbf{y} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution to the homogeneous equation. This is because $A(\mathbf{y} + \mathbf{v}_h) = A\mathbf{y} + A\mathbf{v}_h = \mathbf{b} + \mathbf{0} = \mathbf{b}$.

Problem 2.

Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix}$.

a) (20 points) Find a solution to the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \begin{bmatrix} 3 \\ -1 \\ -6 \\ -2 \end{bmatrix}$.

Solution: Use the given vectors to make the matrix $A = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix}$, then form the augmented matrix by appending the right-hand side vector, then row reduce:

$$\begin{bmatrix} 3 & 0 & 3 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & 2 & -6 \\ -1 & -1 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus we have the equations $x_1 + x_3 = 1$, $x_2 = -1$, $x_3 = -2$. Using back substitution,

$x_1 = 1 - (-2) = 3$. So the solution is $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$.

b) (10 points) Is the solution you found above unique? Why or why not?
Solution: Yes, the solution is unique because every column of the row-reduction of A has a pivot position, and thus there are no free variables.

Problem 3. (15 points each)

Let $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$.

a) Find A^{-1}.

Solution: We consider the augmented matrix $[A|I_3]$ and row reduce until the left-hand side becomes the identity:

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

Thus the inverse is the matrix on the right side, $A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$.

b) Use A^{-1} to find the solution to $A\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$

Solution: If $A\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$, then $\mathbf{x} = A^{-1} \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 14 \\ 18 \\ 8 \end{bmatrix}$.

Problem 4. (10 points each)

a) Give an example of a 4×3 matrix A such that the solution set to $A\mathbf{x} = \mathbf{0}$ is a line.

Solution: We want a matrix where the homogeneous equation has one free variable, and it is easiest to use matrices made of zeros and ones, in echelon form, so let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then the solution set to $A\mathbf{x} = \mathbf{0}$ is all $\mathbf{x} \in \mathbb{R}^3$ such that $x_1 = x_2 = 0$. Let $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Then the solution set is span$\{\mathbf{u}\}$, which is a line. In parametric form, the solution set is all $\mathbf{x}$ of the form $\mathbf{x} = s\mathbf{u}$, where s is any real number.

b) Give an example of a 4×3 matrix A such that $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ of the form $\mathbf{b} = \begin{bmatrix} b_1 \\ 0 \\ b_3 \\ b_4 \end{bmatrix}$ and say why your example works.

Solution: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then it is clear that for a $\mathbf{b}$ of the form given, the solution is $x_1 = b_1$, $x_2 = b_3$, $x_3 = b_4$ (so solutions exist). The solution is unique because there is a pivot in every row of A, and thus there are no free variables in the equation.