12.3: The Dot Product

Some Formulas and Properties to Remember:

The **Dot product** $\mathbf{v} \cdot \mathbf{w}$ of TWO VECTORS

$$\mathbf{v} = \langle a, b, c \rangle, \quad \langle d, e, f \rangle$$

is the SCALAR defined by

$$\mathbf{v} \cdot \mathbf{w} = ad + be + cf$$

In other words, you multiply each corresponding component together, and then you add them up.

Properties of the Dot Product

1. $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = \mathbf{0}$
2. **Commutativity**: $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
3. **Pulling out scalars:** $(\lambda \mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (\lambda \mathbf{w}) = \lambda(\mathbf{v} \cdot \mathbf{w})$
4. **Distributive Law:** $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
   $(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$
5. $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$

ALTERNATIVELY! We have another expression for the dot product:

$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||||\mathbf{w}|| \cos(\theta)$$

where θ is the angle between the two vectors. So we get that the angle between two (non-zero) vectors is:

$$\theta = \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}||||\mathbf{w}||}\right)$$

Important Notes:

1. The dot product is **commutative**, so ORDER DOES NOT MATTER. I.e. $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
2. The dot product takes two VECTORS as input and returns a SCALAR. So things like $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$ are generally impossible.
3. **REMEMBER THIS FACT:** For any non-zero vectors $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{v} \cdot \mathbf{w} = \mathbf{0} \Leftrightarrow$ The angle θ between $\mathbf{v}$ and $\mathbf{w}$ is $90°$ or $\frac{\pi}{2}$ radians. **In other words, the dot product of two non-zero vectors is 0 if and only if the vectors are perpendicular to each other.**

12.4: The Cross Product

The **Cross Product** $\mathbf{v} \times \mathbf{w}$ of TWO VECTORS

$$\mathbf{v} = \langle a, b, c \rangle, \quad \langle d, e, f \rangle$$

is the VECTOR defined by:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix} = i \begin{vmatrix} b & c \\ e & f \end{vmatrix} - j \begin{vmatrix} a & c \\ d & f \end{vmatrix} + k \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$
$$= (bf - ce)\mathbf{i} - (af - cd)\mathbf{j} + (ae - bd)\mathbf{k} = \langle bf - ce, cd - af, ae - bd \rangle$$

Some Formulas and Properties to Remember:

Properties of the Cross Product

1. $\mathbf{v} \times \mathbf{v} = \mathbf{0}$
2. **ANTI-Commutativity**: $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$
3. **Pulling out scalars:** $(\lambda \mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (\lambda \mathbf{w}) = \lambda(\mathbf{v} \times \mathbf{w})$
4. **Distributive Law:** $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
   $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
5. $\mathbf{v} \times \mathbf{w} = 0$ IF AND ONLY IF $\mathbf{v} = \lambda \mathbf{w}$ for some scalar λ OR $\mathbf{v}$ or $\mathbf{w}$ are $\mathbf{0}$

Geometric Description of Cross Product

The cross product is the unique vector following three properties

1. It is orthogonal to $\mathbf{v}$ and $\mathbf{w}$
2. $||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}|| ||\mathbf{w}|| \sin(\theta)$ (θ is the angle between the two vectors and $0 \leq \theta \leq \pi$)
3. The vectors $\{\mathbf{v}, \mathbf{w}, \mathbf{v} \times \mathbf{w}\}$ form a right-handed system

NOTE: The dot product has a similar formula, but with cosine instead of sine. DO NOT GET THESE FORMULAS CONFUSED.

What is a right handed system?

Let's say $\mathbf{u} = \mathbf{v} \times \mathbf{w}$. Then look at the (smaller) angle between $\mathbf{v}$ and $\mathbf{w}$. If you can get from $\mathbf{v}$ to $\mathbf{w}$ **counter-clockwise**, then the vector $\mathbf{u}$ points **towards you**. If you can get from $\mathbf{v}$ to $\mathbf{w}$ **clockwise**, then the vector $\mathbf{u}$ points **away from you**. Then the three vectors form what's called a right-handed system.

Important Notes:

1. The cross product is **anti-commutative**, so ORDER MATTERS.
   In other words $\mathbf{v} \times \mathbf{w} \neq \mathbf{w} \times \mathbf{v}$. Instead, $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$

2. The dot product takes two VECTORS as input and returns a VECTOR. So we can do $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

3. We can ALSO find the triple product: $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$,
   but we CANNOT do this: $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) \times (\mathbf{v} \cdot \mathbf{w})$ because when you distribute, you end up trying to cross two scalars
   and we CANNOT do this: $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}$, because then you would be trying to cross a scalar with a vector