

14.1: Multivariable Functions

Example function: $z = g(x, y) = x^2 + y^2$

Types of traces:

1. Vertical Trace in the plane $x = a$. Set $x = a$ and then see the function. In the example, we have: $z = a^2 + y^2$, which is a parabola
2. Vertical Trace in the plane $y = b$. Set $y = b$ and then see the function. In the example, we have: $z = x^2 + b^2$, which is also a parabola
3. Horizontal Trace in the plane $z = c$. Set $z = c$ and then see the function. In the example, we have: $c = x^2 + y^2$, which is a circle

Tips on Domain and Range:

1. Look at the parent function's domain in single variable. So for example: $\arccos(rs)$. $\arccos(u)$ is defined on $-1 \leq u \leq 1$, so then $-1 \leq rs \leq 1$. This is the area bounded by the hyperbolas $rs = 1$ and $rs = -1$.
2. Be careful, the multivariable erms may limit the domain. Example: $\arccos(\sqrt{rs})$ can only have $0 \leq rs \leq 1$.
3. The range will be whatever values the function is able to take using the domain. So if $\arccos rs$ has domain $[-1, 1]$, then the range is $[0, \pi]$, because those are the values of the range of $\arccos(u)$.

14.2: Continuity and Limits in Several Variables

Three things you can do to find limit:

1) Plug in the variables

If you want the limit at point (a, b) , and the function is continuous at (a, b) , then you just plug in the values of (a, b) into the function. This generally means that you don't get $\frac{0}{0}$, divide by 0, or take the square root of a negative number or other weird things anywhere.

Example: $\lim_{(x,y) \rightarrow (0,0)} e^{x+y^2} = e^{0+0^2} = e^0 = 1$

2) Try to make a sandwich/ use squeeze theorem

Basically, you might have a function that you can't plug in the numbers for without getting $\frac{0}{0}$, dividing by 0, or taking the square root of a negative number, or other weird things anywhere. In which case, your next best guess is to make your function easier to deal with. You want to use the **Squeeze Theorem** to trap weird functions into easy, nice functions. If

those easy, nice functions approach the same limit, then the weird function, trapped between them, must also approach that limit.

Example:

$$\lim_{(x,y) \rightarrow (0,0)} x^4 \sin\left(\frac{1}{x^2 + |y|}\right)$$

We have that $-1 \leq \sin\left(\frac{1}{x^2 + |y|}\right) \leq 1$, and we can use this to make the function easier. Since we have that, we can multiple everything by x^4 and get:

$$-x^4 \leq x^4 \sin\left(\frac{1}{x^2 + |y|}\right) \leq x^4$$

Next, we take the limits:

$$0 = \lim_{(x,y) \rightarrow (0,0)} -x^4 \leq \lim_{(x,y) \rightarrow (0,0)} x^4 \sin\left(\frac{1}{x^2 + |y|}\right) \leq \lim_{(x,y) \rightarrow (0,0)} x^4 = 0$$

So the limit of our example function is going to be stuck between the two limits of the simpler functions. But those limits are both 0. SO **by the Squeeze Theorem** we get:

$$\lim_{(x,y) \rightarrow (0,0)} x^4 \sin\left(\frac{1}{x^2 + |y|}\right) = 0$$

TIPS for trying to find functions to use the squeeze theorem

1. $\sin(\text{something})$ and $\cos(\text{something})$ are ALWAYS between 1 and -1 .
2. $|y| = \sqrt{y^2} \leq \sqrt{x^2 + y^2}$ so $\frac{y}{\sqrt{x^2 + y^2}}$ is always between 1 and -1
3. When you're stuck, see if you can get rid of the ugliest part of the function somehow. This doesn't always work.

3) Prove the limit does not exist

This one is generally the hardest of the three. You basically want to prove the limit does not exist. In single variable, you could do this by proving that the limit from the left and the limit from the right aren't equal. In multivariable, you just need to prove that the limit isn't the same for any two directions.

Example:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + y^2}$$

One way to do this is go from the x direction (basically set $y = 0$ and find the limit), and then the y direction (basically set $x = 0$ and find the limit). Show that something isn't right.

x direction:

$$\lim_{(x,0) \rightarrow (0,0)} \frac{x^2}{x^2 + y^2} = \lim_{(x,0) \rightarrow (0,0)} \frac{x^2}{x^2 + 0^2} = \lim_{(x,0) \rightarrow (0,0)} \frac{x^2}{x^2} = 1$$

y direction:

$$\lim_{(0,y) \rightarrow (0,0)} \frac{x^2}{x^2 + y^2} = \lim_{(0,y) \rightarrow (0,0)} \frac{0^2}{0^2 + y^2} = \lim_{(0,y) \rightarrow (0,0)} \frac{0}{y^2} = 0$$

The limit in the x direction and the limit in the y direction are not equal, so then the limit does not exist.

TIPS for proving the limit does not exist

1. Take the limit in the x direction by setting $y = 0$ and the limit in the y direction by setting $x = 0$.
2. Take the limit along the line $y = x$, by setting $y = x$ in the limit. Things should cancel out to make things easier. Then take the limit in the x or y direction and see what happens.

14.3: Partial Derivatives

Partial derivatives are a lot like derivatives in one dimension. The difference is that, in multivariable calculus, you take derivatives in multiple dimensions. Partial derivatives are derivatives in multivariable functions, BUT WITH RESPECT TO ONE VARIABLE. **You hold every other variable constant.** Single variable derivatives are the rate of change in one dimension. **Multi variable partial derivatives are the rates of change with respect to each variable separately.**

Single Variable Derivative (Review):

$$\frac{d}{dx} f(x) = f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Multi-Variable Partial Derivatives:

Example equation:

$$g(x, y) = x^2 y^3 + \ln x + \sin y$$

IMPORTANT NOTE: When you take the partial derivative with respect to a variable, **YOU HOLD ALL OTHER VARIABLE CONSTANT.** Do not, Do Not, DO NOT forget that you are treating them as constants, and DO NOT forget that they are there. That is one of the most common ways to lose points on an exam.

Partial Derivative with respect to x:

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Example equation partial with respect to x (PRETEND y IS CONSTANT):

$$g_x(x, y) = 2xy^3 + \frac{1}{x} \quad \text{Note: } y \text{ is constant here, so } \sin y \text{ is constant, and its derivative is } 0$$

Partial Derivative with respect to y:

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Example equation partial with respect to y (PRETEND x IS CONSTANT):

$$g_y(x, y) = 3x^2y^2 + \cos y \quad \text{Note: } x \text{ is constant here, so } \ln x \text{ is constant, and its derivative is 0}$$

Second Order Partial Derivatives:

$$\frac{\partial^2}{\partial y^2} f(x, y) = f_{yy}(x, y) = \frac{\partial}{\partial y} \left(\frac{\partial}{\partial y} f(x, y) \right)$$

$$\frac{\partial^2}{\partial x^2} f(x, y) = f_{xx}(x, y) = \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} f(x, y) \right)$$

Example equation second order partial with respect to x (PRETEND y IS CONSTANT):

$$g_{xx}(x, y) = \frac{\partial}{\partial x} \left(2xy^3 + \frac{1}{x} \right) = 2y^3 - \frac{1}{x^2}$$

Example equation second order partial with respect to y (PRETEND x IS CONSTANT):

$$g_{yy}(x, y) = \frac{\partial}{\partial y} \left(3x^2y^2 + \cos y \right) = 6x^2y - \sin y$$

Second Order MIXED Partial Derivatives:

$$\frac{\partial^2}{\partial x \partial y} f(x, y) = f_{xy}(x, y) = \frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} f(x, y) \right)$$

$$\frac{\partial^2}{\partial y \partial x} f(x, y) = f_{yx}(x, y) = \frac{\partial}{\partial y} \left(\frac{\partial}{\partial x} f(x, y) \right)$$

Example equation second order partial with respect to y THEN x :

$$g_{xy}(x, y) = \frac{\partial}{\partial x} \left(3x^2y^2 + \cos y \right) = 6xy^2$$

Example equation second order mixed partial with respect to x THEN y :

$$g_{yx}(x, y) = \frac{\partial}{\partial y} \left(2xy^3 + \frac{1}{x} \right) = 6xy^2$$

Notice that they're equal? That's because of **Clairaut's Theorem**. It basically states **you can take the partial derivatives in any order as long as the partials are continuous**.