Announcements

1. There will be a Review Session. It is planned to be on Sunday from 3PM to 5PM again. I'll let you know the location later when I find out where it is.
2. There will also be a practice midterm and solutions, but I have not written it yet

14.4: Tangent Planes in 3 Dimensions and Linearizations

The idea for this section is that the plane tangent to a point on a nice surface is CLOSE to the surface near the point where it is tangent to. **Linearization of** $z = f(x, y)$ **at** (a, b) is:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

This is exactly the point (x, y, z) on the tangent plane that corresponds to $(x, y, f(x, y))$. The equation for the tangent plane at the surface
We can ALSO write it like this, with $x = a + h$ and $y = b + k$:

$$L(x, y) = L(a + h, b + k) = f(a, b) + f_x(a, b)h + f_y(a, b)k$$

The **linear approximation** of a function at point (x, y) is the value of the linearization at that point. So we have:

$$f(x, y) \approx L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

(NOTE: You want (a, b) to be close to (x, y) to get a good approximation)

We can also do linear approximations in 3 variables:

$$f(x, y, z) \approx L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

(NOTE: You want (a, b, c) to be close to (x, y, z) to get a good approximation)

We can also use the linear approximation to **estimate the change in f**, denoted by Δf

$$\Delta f \approx f_x(a, b)(x - a) + f_y(a, b)(y - b) = f_x(a, b)\Delta x + f_y(a, b)\Delta y$$

We can also use linear approximation to find the **differential**:

$$df = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

We can also find differentials in 3 variables:

$$df = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz$$

The differential df **gives you the exact change in the tangent plane at the point** (x, y) **when you change** x **and** y **from the point** (x, y)**. The change in the function,**

Δf gives you an approximation of the change of the value of the function from some point (a, b). Honestly this section is basically the same formula repeated like five times, but changed slightly every time.

Example Problem:

Use linear approximation to estimate the value of:

$$\arctan(\ln(1.02)) + \cos(-0.01)$$

Step 1: Figure out a function to use that would correspond to this mess.

The general method of doing this is to replace every unique number with a variable. So replacing 1.02 with x and replacing -0.01 with y, we get:

$$f(x, y) = \arctan(\ln(x)) + \cos(y)$$

Step 2: Find an easy point to evaluate.

The general method of doing this would be taking the integers CLOSEST to the values you are trying to approximate. So if you're trying to find arctan(ln(1.02)), then arctan(ln(1)) would be easy to find, and it's close to the value you're trying to evaluate. If you're trying to find cos(−0.01), then cos(0) would be easy to find, and it's close to the value you're trying to evaluate. So set $a = 1$, $b = 0$.

Step 3: Find the partial derivatives, and evaluate f, f_x, and f_y at (a, b).

You need this to plug it into the formula above.

$$f_x(x, y) = \frac{\frac{\partial}{\partial x}(\ln(x))}{(\ln(x))^2 + 1} + \frac{\partial}{\partial x}(\cos(y)) = \frac{1}{x\big((\ln(x))^2 + 1\big)} \;\Bigg|\; f_x(a, b) = f_x(1, 0) = \frac{1}{1\big((\ln(1))^2 + 1\big)} = 1$$

$$f_y(x, y) = \frac{\partial}{\partial y}(\arctan(\ln(x)) + \frac{\partial}{\partial y}(\cos(y)) = -\sin(y) \;\Bigg|\; f_y(a, b) = f_y(1, 0) = -\sin(0) = 0$$

$$f(a, b) = f(1, 0) = \arctan(\ln(1)) + \cos(0) = \arctan(0) + \cos(0) = 0 + 1 = 1$$

Plug and chug into the formula:

$$\begin{aligned} f(x, y) &= f(1.02, -0.01) \approx L(1.02, -0.01) \\ &= f(1, 0) + f_x(1, 0)(1.02 - 1) + f_y(1, 0)(-0.01 - 0) \\ &= 1 + 1(0.02) + 0(-0.01) = \boxed{1.02} \end{aligned}$$