Announcements:

1. Midterm 2 is tomorrow! It covers setions 13.1-13.3, 13.5,14.1-14.5. There will be NO GRAPHING/MATCHING problems, and NO DOMAIN/RANGE problems. 13.4, 13.6, and 14.6 will NOT be on the midterm.
2. The midterm is 4 problems, 20 points each
3. Yes, you can use a cheat sheet, 8.5, double-sided, HANDWRITTEN
4. PLEASE write your name, PID, section, and version number CLEARLY on the cover.

14.5: The Gradient and Directional Derivatives

The gradient of f is a VECTOR FUNCTION given by:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \text{ (in 3-D, for } f(x,y,z)) \quad \Bigg| \quad \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \text{ (in 2-D, for } f(x,y))$$

At a point (a, b, c), the function would give us a constant vector, that gives us the direction of **greatest increase from the point**. One of the implications of this is that the gradient will be **NORMAL to the level curves** of a function of two variables $f(x, y)$ and **NORMAL to the level surfaces** of a function of three variables $f(x, y, z)$. The second part is important because if you have some level surface $f(x, y, z) = c$, then the **EQUATION OF THE TANGENT PLANE AT POINT** (a, b, c) **is given by**:

$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0$$

where $\langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c)\rangle$ is the gradient at point (a, b, c). Recall: you can get the equation of a tangent plane with a NORMAL VECTOR and a POINT. The gradient is a normal vector to a level surface at a point, SO it can be your normal vector for your tangent plane.

The directional derivative at point $P = (a, b, c)$ and in the direction of some vector $\mathbf{v}$ is a NUMBER given by the formula:

$$D_{\mathbf{v}}(a, b, c) = \nabla f(a, b, c) \cdot \mathbf{e}_{\mathbf{v}}$$

Where $\mathbf{e}_{\mathbf{v}}$ is the unit vector of $\mathbf{v}$. The directional derivative gives us a number indicating the amount of CHANGE that occurs in the direction we are looking at. The closer the direction is to the gradient, the greater the value of the directional derivative. If the direction is closer to the opposite direction to the gradient, then it is going to be negative, and it is going to be a direction of DECREASE.

We can ALSO write the directional derivative using the alternate formula for the dot product:

$$D_{\mathbf{v}}(a, b, c) = \nabla f(a, b, c) \cdot \mathbf{e}_{\mathbf{v}} = \|\nabla f(a, b, c)\|\|\mathbf{e}_{\mathbf{v}}\| \cos\theta = \|\nabla f(a, b, c)\| \cos\theta$$

Where θ is the angle between the gradient and the vector that represents the direction you are finding the directional derivative for.

Other notes in this section:

1. the directional derivative in the direction of a vector that is TANGENT to the level curve at the point is 0. That is because, along a level curve, the value of the function is CONSTANT, and therefore,

2. The gradient is going to be NORMAL/PERPEDICULAR to the level curve/surface. It represents the direction of greatest INCREASE, so it would be as FAR from the level curve as possible.

14.6: The Chain Rule

Chain Rule

Chain rule for paths (14.5):

$$\frac{d}{dt}f(c(t)) = \nabla f(c(t)) \cdot c'(t)$$

This is something that gives you the rate of change in the direction of a particle moving along the path $c(t)$, when it is going at a specific velocity. However, you more often want to find the directional derivative.

Chain Rule Implicit Differentiation

Chain rule: With $x(r,s)$, $y(r,s)$, $z(r,s)$, we have for some function $f(x,y,z)$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$
$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

And we endup with a derivative that takes the rates of change of each of the dependent variables x, y, and z, multiplied by the rate of change of those dependent variales with respect to the independent ones. The sum is a TOTAL of the rates of change of each of the function's variables with respect to a variable NOT directly tied to the function f.
We can extend this to implicitly derive a function $z = f(x,y)$, by finding some function $f(x,y,z) = 0$, and differentiating. This results in:

$$\begin{aligned} 0 &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} \implies \frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} \end{aligned}$$

We apply chain rule with respect to x, and then solve for $\frac{\partial z}{\partial x}$. y does NOT depend on x.