1 Position, Velocity, Acceleration, and Newton's Law

A drunk **10kg** space cat with a rocket pack is flying dangerously hrough space, annoying alien scientists that are also trying to fly through space. The alien scientists eventually manage to analyze the space cat's rocket pack, and find that the force exerted by the rocket pack is given by the following formula:

$$\mathbf{F}(t) = \left\langle \frac{-20t}{(t^2+1)^2}, 60t, 20 \right\rangle$$

At a particular moment of time, they also manage to analyze and determine the space cat's exact **velocity** $\langle 1, 10, -2 \rangle$ and **position** $\langle 1, 31, 1 \rangle$ at that point in time. They set these values equal to the **initial velocity** and **initial position**

a) Find the acceleration, velocity, and position of the drunk space cat as functions of time t

Solution:

Firstly, note that in this part of the problem, we are looking for **THREE FUNCTIONS**. Given that the force is expressed in terms of a vector-valued function, the accleration, velocity, and position would ALSO be **VECTOR-VALUED FUNCTIONS**.

Step 1: Find the acceleration using Newton's Law

We start with the acceleration. We aren't given this, but we are given FORCE, and MASS. So we can apply Newton's Law (FORCE = MASS x ACCELERATION):

$$\mathbf{F}(t) = m\mathbf{a}(t) \implies \mathbf{a}(t) = \frac{1}{m}\mathbf{F}(t) = \frac{1}{10}\left\langle \frac{-20t}{(t^2+1)^2}, 60t, 20 \right\rangle = \boxed{\left\langle \frac{-2t}{(t^2+1)^2}, 6t, 2 \right\rangle = \mathbf{a}(t)}$$

NOTE: Sometimes you ARE given the acceleration function. PLEASE READ THE PROBLEM CAREFULLY.

Step 2: Find the Velocity function

Next, we have to find velocity and position. **The velocity is the integral of the acceleration, and the position is the integral of the velocity**. PLEASE MAKE SURE TO FIND THE CONSTANTS OF INTEGRATION.

$$\mathbf{v}(t) = \int \mathbf{a}(t) = \left\langle \int \frac{-2t}{(t^2+1)^2}dt, \int 6t dt, \int 2dt \right\rangle = \left\langle \frac{1}{t^2+1} + c_1, 3t^2 + c_2, 2t + c_3 \right\rangle$$

Note: For velocity, to take the integral of the acceleration's first component, I used u-substution with $u = t^2 + 1$. Then you use the power rule. For the other two functions, you use the power rule. We then find the constants of integration (c_1,c_2, and c_3) by finding $\mathbf{v}(0)$ and setting that equal to the initial velocity.

OTHER NOTE: The three constants of integration are NOT NECESSARILY THE SAME for each component. That is because those constants of integration act independently of each other.

$$\begin{aligned}
\mathbf{v}(0) &= \left\langle \frac{1}{0^2+1} + c_1, 3(0)^2 + c_2, 2(0) + c_3 \right\rangle = \langle 1 + c_1, c_2, c_3 \rangle = \langle 1, 10, -2 \rangle \\
&\implies \langle c_1, c_2, c_3 \rangle = \langle 0, 10, -2 \rangle \\
&\implies \boxed{\mathbf{v}(t) = \left\langle \frac{1}{t^2+1}, 3t^2 + 10, 2t - 2 \right\rangle}
\end{aligned}$$

NOTE: Here we see that the constants of integration ARE NOT ALWAYS THE SAME AS THE INITIAL VELOCITY/POSITION. BE CAREFUL.

Step 3: Find the Position function

$$\begin{aligned}
\mathbf{r}(t) = \int \mathbf{v}(t) &= \left\langle \int \frac{1}{t^2+1} dt, \int 3t^2 + 10dt, \int 2t - 2dt \right\rangle \\
&= \left\langle \arctan(t) + k_1, t^3 + 10t + k_2, t^2 - 2t + k_3 \right\rangle
\end{aligned}$$

NOTE: The three constants of integration are NOT NECESSARILY THE SAME for each component. That is because those constants of integration act independently of each other. We then find the constants of integration (k_1,k_2, and k_3) by finding $\mathbf{r}(0)$ and setting that equal to the initial position.

$$\begin{aligned}
\mathbf{r}(0) &= \left\langle \arctan(0) + k_1, 0^3 + 10(0) + k_2, 0^2 - 2(0) + k_3 \right\rangle = \langle k_1, k_2, k_3 \rangle = \langle 1, 31, 1 \rangle \\
&\implies \langle k_1, k_2, k_3 \rangle = \langle 1, 31, 1 \rangle \\
&\implies \boxed{\mathbf{r}(t) = \left\langle \arctan(t) + 1, t^3 + 10t + 31, t^2 - 2t + 1 \right\rangle}
\end{aligned}$$

b) The alien scientists also prepared an electric field trap for the drunk cat over the entire xy-plane (z = 0). Does the cat ever reach this electric field? If so, find the time at which the cat reaches the electric field, as well as its position, velocity, speed, and acceleration at that time.

Solution:

Note that this part of the problem asks for (potentially) **FIVE THINGS**: **(1)** whether the drunken cat hits the electric field trap (yes/no), and if it does, **(2)** the TIME that it hits the electric field trap (expressed as a NUMBER), and **(3)** the position, **(4)** the velocity, and **(5)** the acceleration at that point in time (expressed as VECTORS).

Step 1: Check to see if the cat even gets zapped

First, check that the z-component of the position, $z(t) = t^2 - 2t + 1$ can actually be 0 at some point in time. If it does, that means that the cat'sposition will be in the xy-plane, exactly where the trap is set.

$t^2 - 2t + 1 = (t - 1)^2 = 0$ at $t = 1$. **So the drunk cat does reach the electric field trap, and it does so at time** $\boxed{t = 1}$.

Step 2: Find the acceleration, velocity, and position at the time t you found

Basically, you just plug in $t = 1$ into the acceleration, velocity, and position functions.

$$\mathbf{a}(1) = \left\langle \frac{-2(1)}{(1^2+1)^2}, 6(1), 2 \right\rangle = \boxed{\langle -\tfrac{1}{2}, 6, 2 \rangle}$$

$$\mathbf{v}(1) = \left\langle \frac{1}{1^2+1}, 3(1)^2 + 10, 2(1) - 2 \right\rangle = \boxed{\langle \tfrac{1}{2}, 13, 0 \rangle}$$

$$\mathbf{r}(1) = \left\langle \arctan(1) + 1, 1^3 + 10(1) + 31, 1^2 - 2(1) + 1 \right\rangle = \boxed{\langle \tfrac{\pi}{4} + 1, 42, 0 \rangle}$$

Step 3: Find the speed at time t

The speed at time $t = 1$ is just the magnitude of the velocity at time $t = 1$. SPEED IS A NON-NEGATIVE NUMBER

$$\|\mathbf{v}(1)\| = \sqrt{\left(\tfrac{1}{2}\right)^2 + 13^2 + 0^2} = \boxed{\sqrt{169.25}}$$

c) After dealing with the drunk space cat, the alien scientists realized they missed it dearly, and decided to create a flying robot cat to fill the void in their lives. Find the velocity and acceleration of the cat (as functions in terms of t), given that the robot cat's position is expressed as:

$$\mathbf{r}(t) = \left\langle \arctan(t) + 21, \ln t + 13, \sin t + \pi \right\rangle$$

Solution:

This part of the problem asks for **TWO THINGS**. The **velocity function** and the **acceleration function**. These are **VECTOR-VALUED FUNCTIONS**. The velocity function is the derivative of the position function, and the acceleration is the derivatve of the velocity function. SO, the solution is:

$$\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t) = \left\langle \frac{d}{dt}\Big(\arctan(t) + 21\Big), \frac{d}{dt}\Big(\ln t + 13\Big), \frac{d}{dt}\Big(\sin t + \pi\Big) \right\rangle = \boxed{\left\langle \frac{1}{t^2+1}, \frac{1}{t}, \cos t \right\rangle}$$

$$\mathbf{a}(t) = \frac{d}{dt}\mathbf{v}(t) = \left\langle \frac{d}{dt}\Big(\frac{1}{t^2+1}\Big), \frac{d}{dt}\Big(\frac{1}{t}\Big), \frac{d}{dt}\Big(\cos t\Big) \right\rangle = \boxed{\left\langle -\frac{2t}{(t^2+1)^2}, -\frac{1}{t^2}, -\sin t \right\rangle}$$

2 Limits and Continuity

Evaluate the following limits if they exist, otherwise, show that they do not exist.

a)

$$\lim_{(x,y)\to(0,0)} e^{x+y^2}$$

Solution:

In this case, you can just plug in the point and evaluate. You get a number that isn't the square root of a negative number $\frac{0}{0}$, or dividing by 0, or some other weird thing. The function is continuous at this point, and therefore, the limit is just the function value at that point.

$$\lim_{(x,y)\to(0,0)} e^{x+y^2} = e^{0+0^2} = e^0 = \boxed{1}$$

b)

$$\lim_{(x,y)\to(0,0)} x^4 \sin\left(\frac{1}{x^2+|y|}\right)$$

Solution:

You might have a function that you can't plug in the numbers for without getting $\frac{0}{0}$, dividing by 0, or taking the square root of a negative number, or other weird things anywhere. In which case, your next best guess is to make your function easier to deal with. You want to use the **Squeeze Theorem** to trap weird functions between easy, nice functions. If those easy, nice functions approach the same limit, then the weird function, trapped between them, must also approach that limit.

$$\lim_{(x,y)\to(0,0)} x^4 \sin\left(\frac{1}{x^2+|y|}\right)$$

We have that $-1 \leq \sin(\frac{1}{x^2+|y|}) \leq 1$, and we can use this to make the function easier. Since we have that, we can multiple everything by x^4 and get:

$$-x^4 \leq x^4 \sin\left(\frac{1}{x^2+|y|}\right) \leq x^4$$

Next, we take the limits:

$$0 = \lim_{(x,y)\to(0,0)} -x^4 \leq \lim_{(x,y)\to(0,0)} x^4 \sin\left(\frac{1}{x^2+|y|}\right) \leq \lim_{(x,y)\to(0,0)} x^4 = 0$$

So the limit of our example function is going to be stuck between the two limits of the simpler functions. But those limits are both 0. SO **by the Squeeze Theorem** we get:

$$\boxed{\lim_{(x,y)\to(0,0)} x^4 \sin\left(\frac{1}{x^2+|y|}\right) = 0}$$

TIPS for trying to find functions to use the squeze theorem

1. sin(something) and cos(something) are ALWAYS between 1 and -1.
2. $|y| = \sqrt{y^2} \leq \sqrt{x^2+y^2}$ so $\frac{y}{\sqrt{x^2+y^2}}$ is always between 1 and -1
3. When you're stuck, see if you can get rid of the ugliest part of the function somehow. This doesn't always work.

c)

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$$

Solution:

This one is generally the hardest of the three. You basically want to prove the limit does not exist. In single variable, you could do this by proving that the limit from the left and the limit from the right aren't equal. In multivariable, you just need to prove that the limit isn't the same for any two directions.

One way to do this is go from the x direction (basically set $y = 0$ and find the limit), and then the y direction (basically set $x = 0$ and find the limit). Show that something isn't right (i.e. their limits are different).

x direction:

$$\lim_{(x,0)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2+0^2} = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$$

y direction:

$$\lim_{(0,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{0^2}{0^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{y^2} = 0$$

The limit in the x direction and the limit in the y direction are not equal, so then **the limit does not exist**.

TIPS for proving the limit does not exist

1. Take the limit in the x direction by setting $y = 0$ and the limit in the y direction by setting $x = 0$.
2. Take the limit along the line $y = x$, by setting $y = x$ in the limit. Things should cancel out to make things easier. Then take the limit in the x or y direction and see what happens.

3 Linear Approximations

Use linear approximation to estimate the value of:

$$\arctan(\ln(1.02)) + 1.02\sin(-0.01) + 42$$

Solution:

You want to use the formula for the linearization of a function that uses point (a, b) to estimate the value of the function at point (x, y), given by:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Step 1: Figure out a function to use that would correspond to this mess.

The general method of doing this is to replace every unique (weird) number with a variable. So replacing 1.02 with x and replacing -0.01 with y, we get:

$$f(x, y) = \arctan(\ln(x)) + x\sin(y) + 42$$

NOTE: I did not replace 42 with a variable. That's because it stands on its own and is not connected to any parent function like sin, ln, or arctan, etc.

IMPORTANT NOTE: Sometimes the function is given to you. In which case, you can obviously skip this step and move onto the next one.

Step 2: Find an easy point to evaluate.

The general method of doing this would be taking the integers CLOSEST to the values you are trying to approximate. So if you're trying to find $\arctan(\ln(1.02))$, then $\arctan(\ln(1))$ would be easy to find, and it's close to the value you're trying to evaluate. If you're trying to find $1.02\sin(-0.01)$, then $1\sin(0)$ would be easy to find, and it's close to the value you're trying to evaluate. So set $a = 1$, $b = 0$. **Your point for the linearization is** $(a, b) = (1, 0)$.

Step 3: Find the partial derivatives, and evaluate f, f_x, and f_y at (a, b).

You need this to plug it into the formula above.

$$\begin{aligned} f_x(x,y) &= \frac{\frac{\partial}{\partial x}(\ln(x))}{(\ln(x))^2+1} + \frac{\partial}{\partial x}(x\sin(y)) + \frac{\partial}{\partial x}(42) = \frac{1}{x\big((\ln(x))^2+1\big)} + \sin(y) \\ f_x(a,b) &= f_x(1,0) = \frac{1}{1\big((\ln(1))^2+1\big)} + \sin(0) = 1 + 0 = 1 \end{aligned}$$

$$\begin{aligned} f_y(x,y) &= \frac{\partial}{\partial y}(\arctan(\ln(x)) + \frac{\partial}{\partial y}(x\sin(y)) + \frac{\partial}{\partial y}(42) = x\cos(y) \\ f_y(a,b) &= f_y(1,0) = 1\cos(0) = 1 \end{aligned}$$

$$f(a,b) = f(1,0) = \arctan(\ln(1)) + 1\sin(0) + 42 = \arctan(0) + 1\sin(0) + 42 = 0 + 0 + 42 = 42$$

Step 4: Plug and chug into the formula

$$\begin{aligned} f(x,y) = f(1.02,-0.01) &\approx L(1.02,-0.01) \\ &= f(1,0) + f_x(1,0)(1.02-1) + f_y(1,0)(-0.01-0)) \\ &= 42 + 1(0.02) + 1(-0.01-0) = \boxed{42.01} \end{aligned}$$

NOTE: The answer to this question WILL BE A NUMBER OF SOME KIND. Do NOT give the answer in the form of the linearization formula or some other kind of formula. READ THE PROBLEM CAREFULLY.

4 Tangent Planes

a)

Find the equation of the tangent plane at point $(a, b) = (1, 0)$ on the surface given by:

$$f(x,y) = \arctan(\ln(x)) + x\sin(y) + 42$$

Solution:

The equation for a tangent plane given a function of x and y at a point (a, b), or at point $(a, b, f(a, b))$ is given by:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Look familiar? That's because this is the same formula used for linear approximations. In fact, tangent plane equations are used in linear approximations, because, close to a point, the value of a function near a point is going to be CLOSE to the value of the tangent plane at that same point.

Step 1: Find the partial derivatives, and evaluate f, f_x, and f_y at (a, b).

You need this to plug it into the formula above.

$$\begin{aligned} f_x(x,y) &= \frac{\frac{\partial}{\partial x}(\ln(x))}{(\ln(x))^2+1} + \frac{\partial}{\partial x}(x\sin(y)) + \frac{\partial}{\partial x}(42) = \frac{1}{x\big((\ln(x))^2+1\big)} + \sin(y) \\ f_x(a,b) &= f_x(1,0) = \frac{1}{1\big((\ln(1))^2+1\big)} + \sin(0) = 1 + 0 = 1 \end{aligned}$$

$$\begin{aligned} f_y(x,y) &= \frac{\partial}{\partial y}(\arctan(\ln(x)) + \frac{\partial}{\partial y}(x\sin(y)) + \frac{\partial}{\partial y}(42) = x\cos(y) \\ f_y(a,b) &= f_y(1,0) = 1\cos(0) = 1 \end{aligned}$$

$$f(a,b) = f(1,0) = \arctan(\ln(1)) + 1\sin(0) + 42 = \arctan(0) + 1\sin(0) + 42 = 0 + 0 + 42 = 42$$

Step 2: Plug and chug into the formula

$$\begin{aligned} z &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)) \\ &= 42 + 1(x-1) + 1y \\ &\boxed{z = 41 + x + y} \end{aligned}$$

This is an acceptable form of the answer. When giving the equation of a plane, note that it is a function of x, y, and z, and DOES NOT DEPEND ON t. READ THE PROBLEM CAREFULLY.

b)

Find the equation of the tangent plane at point (1,1,1), on the surface given implicitly by the function:

$$-\arctan(\ln(z)) = \ln(xy) + x^3y^2 - 1$$

Solution:

The equation for a tangent plane given by an IMPLICIT surface (one that isn't in the form $z = f(x, y)$), at a point (a, b, c) is given by:

$$f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0$$

Of course, for this, you'd need the function $f(x, y, z)$, which you aren't given directly.

Step 1: Find the equation f(x,y,z)

When you're given a surface in this form (z is not all by itself or it is harder to solve for z and take the partial derivatives), you want to make this surface in terms of x, y, and z. So placing all the variable terms on one side, we get:

$$\arctan(\ln(z)) + \ln(xy) + x^3y^2 = 1$$

The terms on the left can be made into your function f. In other words:

$$f(x, y, z) = \arctan(\ln(z)) + \ln(xy) + x^3y^2$$

When $f(x, y, z)$ is set to some constant, you get the LEVEL SURFACE of the function when set to that constant. So our original formula is the level surface of f at $f(x, y, z) = 1$. Thing is, at every point on this level surface, **THE GRADIENT IS NORMAL TO THE LEVEL SURFACE**. Like in the last midterm, to get the equation of a plane, you would need a normal vector and a point. So your normal vector is going to be the gradient at the point you're looking at.

Step 2: Find the gradient

The gradient of f is a VECTOR FUNCTION given by:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

So we find the partial derivatives:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{y}{xy} + 3x^2y^2 = \frac{1}{x} + 3x^2y^2 \\
\frac{\partial f}{\partial y} &= \frac{x}{xy} + 2x^3y = \frac{1}{y} + 2x^3y \\
\frac{\partial f}{\partial z} &= \frac{\frac{1}{z}}{(\ln z)^2 + 1} = \frac{1}{z\big((\ln z)^2 + 1\big)}
\end{aligned}$$

Then we get the gradient of f:

$$\nabla f(x, y, z) = \left\langle \frac{1}{x} + 3x^2y^2, \quad \frac{1}{y} + 2x^3y, \quad \frac{1}{z\big((\ln z)^2 + 1\big)} \right\rangle$$

Step 3: Find the value of the gradient at point $P = (1, 1, 1)$

Basically here we just plug and chug, with $(x, y, z) = (1, 1, 1)$.

$$\begin{aligned}\nabla f(1,1,1) &= \left\langle \frac{1}{1} + 3(1^2)(1^2), \quad \frac{1}{1} + 2(1^3)(1), \quad \frac{1}{1\big((\ln 1)^2 + 1\big)} \right\rangle \\ &= \langle 4, 3, 1 \rangle = \langle f_x(1,1,1), f_y(1,1,1), f_z(1,1,1) \rangle\end{aligned}$$

Step 4: Plug and chug into the formula

Obviously, your point will be the point you were given $P = (1, 1, 1)$ and your normal vector is the gradient at that point, $\langle 4, 3, 1 \rangle$. You can use any of the forms of equations of a plane.

$$\begin{aligned}f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) &= 0 \\ f_x(1,1,1)(x-1) + f_y(1,1,1)(y-1) + f_z(1,1,1)(z-1) &= 0 \\ \boxed{4(x-1) + 3(y-1) + 1(z-1) = 0}\end{aligned}$$

This is an acceptable form of the answer. When giving the equation of a plane, note that it is a function of x, y, and z, and DOES NOT DEPEND ON t. READ THE PROBLEM CAREFULLY.

IMPORTANT NOTE: The first method in part a) should be used for functions in the form $z = f(x, y)$, where z stands by itself. If you have some function of z, YOU SHOULD USE THE SECOND METHOD FROM PART b) instead of solving for z and using the first method.

5 Harmonic Functions

The Laplace operator Δ is defined by $\Delta(f) = f_{xx} + f_{yy}$ for any function $f(x, y)$. A function f is called **harmonic** if $\Delta(f) = 0$. Is the following function harmonic?

$$f(x, y) = xy + \ln(x^2 + y^2)$$

Solution:

This looks fairly intimidating at first, but it's basically just asking you to find the second order partial derivatives (f_{xx} and f_{yy}) and then add them together. If their sum is 0, then you have a harmonic function.

Step 1: Find the second order partial derivative with respect to x

$$f_x = y + \frac{2x}{x^2 + y^2}$$
$$f_{xx} = \frac{(x^2 + y^2)2 - (2x)(2x)}{(x^2 + y^2)^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}$$

NOTE: You're gonna want to remember how to use the chain rule, the product rule, the quotient rule, and the derivative of natural log.

Step 3: Find the second order partial derivative with respect to y

$$f_y = x + \frac{2y}{x^2 + y^2}$$
$$f_{yy} = \frac{(x^2 + y^2)2 - (2y)(2y)}{(x^2 + y^2)^2} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}$$

Step 3: Check to see if the function is harmonic

$$f_{xx} + f_{yy} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2} + \frac{2(x^2 - y^2)}{(x^2 + y^2)^2} = 0 \quad \checkmark$$

So the function is harmonic.

6 Gradients and Directional Derivatives

Find the directional derivative of the function

$$f(x, y, z) = \arctan(\ln(x)) + \ln(xyz) + x^3y^2z$$

In the direction of $\mathbf{v} = \langle 1, -2, 2 \rangle$, at the point $P = (1, 1, 1)$.

Solution:

The directional derivative at point $P = (a, b, c)$ and in the direction of some vector $\mathbf{v}$ is a NUMBER given by the formula:

$$D_{\mathbf{v}}(a, b, c) = \nabla f(a, b, c) \cdot \mathbf{e}_{\mathbf{v}}$$

Where $\mathbf{e}_{\mathbf{v}}$ is the unit vector of $\mathbf{v}$.

Step 1: Find the gradient

The gradient of f is a VECTOR FUNCTION given by:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

So we find the partial derivatives:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\frac{1}{x}}{(\ln x)^2+1} + \frac{yz}{xyz} + 3x^2y^2z = \frac{1}{x\big((\ln x)^2+1\big)} + \frac{1}{x} + 3x^2y^2z \\
\frac{\partial f}{\partial y} &= \frac{xz}{xyz} + 2x^3yz = \frac{1}{y} + 2x^3yz \\
\frac{\partial f}{\partial z} &= \frac{xy}{xyz} + x^3y^2 = \frac{1}{z} + x^3y^2
\end{aligned}$$

Then we get the gradient of f:

$$\nabla f(x,y,z) = \left\langle \frac{1}{x\big((\ln x)^2+1\big)} + \frac{1}{x} + 3x^2y^2z, \quad \frac{1}{y} + 2x^3yz, \quad \frac{1}{z} + x^3y^2 \right\rangle$$

Step 2: Find the value of the gradient at point $P = (1,1,1)$

Basically here we just plug and chug, with $(x,y,z) = (1,1,1)$.

$$\begin{aligned}
\nabla f(1,1,1) &= \left\langle \frac{1}{1\big((\ln 1)^2+1\big)} + \frac{1}{1} + 3(1^2)(1^2)(1), \quad \frac{1}{1} + 2(1^3)(1)(1), \quad \frac{1}{1} + (1^3)(1^2) \right\rangle \\
&= \langle 5,3,2 \rangle
\end{aligned}$$

Step 3: Find the unit vector of v

Simple enough, just divide $\mathbf{v}$ by its magnitude, as before:

$$\mathbf{e}_{\mathbf{v}} = \frac{\langle 1,-2,2\rangle}{\sqrt{1^2+(-2)^2+2^2}} = \frac{\langle 1,-2,2\rangle}{\sqrt{9}} = \frac{\langle 1,-2,2\rangle}{3} = \left\langle \tfrac{1}{3}, -\tfrac{2}{3}, \tfrac{2}{3} \right\rangle$$

DO NOT FORGET THIS STEP.

Step 4: Plug everything into the formula

$$D_{\mathbf{v}}(1,1,1) = \nabla f(1,1,1) \cdot \mathbf{e}_{\mathbf{v}} = \left\langle 5,3,2 \right\rangle \cdot \left\langle \tfrac{1}{3}, -\tfrac{2}{3}, \tfrac{2}{3} \right\rangle = 5\left(\tfrac{1}{3}\right) + 3\left(-\tfrac{2}{3}\right) + 2\left(\tfrac{2}{3}\right) = \boxed{1}$$

NOTE: The directional derivative at a point will be the value of a dot product, WHICH WILL BE A NUMBER. READ THE PROBLEM CAREFULLY.