Note: Fill out those TA Evals (LINK TO TA EVALS THAT YOU SHOULD REALLY REALLY SUPER MEGA ULTRA FILL OUT:
https://academicaffairs.ucsd.edu/Modules/Evals/default.aspx)

1 Dot Product and Cross Product

Let

$$\mathbf{u} = \langle 2, 5, 1 \rangle \qquad \mathbf{v} = \langle 1, 2, -2 \rangle \qquad \mathbf{w} = \langle 1, -3, 2 \rangle$$

a) (Angle Between Two Vectors)

Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

Solution:

The cosine of the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$ is given by the formula:

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

So just plug and chug into the formula:

$$\begin{aligned}
\cos(\theta) &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \\
&= \frac{\langle 2, 5, 1 \rangle \cdot \langle 1, 2, -2 \rangle}{\|\langle 2, 5, 1 \rangle\|\|\langle 1, 2, -2 \rangle\|} \\
&= \frac{2(1) + 5(2) + 1(-2)}{\sqrt{2^2 + 5^2 + 1^2}\sqrt{1^2 + 2^2 + (-2)^2}} \\
&= \frac{10}{\sqrt{30}\sqrt{9}} = \sqrt{\frac{10}{27}}
\end{aligned}$$

To find the ANGLE, you need to take the ARCCOS of the function!!!

$$\cos(\theta) = \sqrt{\frac{10}{27}} \implies \boxed{\theta = \arccos\left(\sqrt{\frac{10}{27}}\right)}$$

b) (Area of a Parallelogram)

Find the area of a parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.

Solution:

The AREA of a prallelogram spanned by two vectors is given by the MAGNITUDE OF THE CROSS PRODUCT. So, we find the cross product $\mathbf{v} \times \mathbf{w}$:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 1 & -3 & 2 \end{vmatrix} = i \begin{vmatrix} 2 & -2 \\ -3 & 2 \end{vmatrix} - j \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix}$$
$$= (4-6)\mathbf{i} - (2-(-2))\mathbf{j} + (-3-2)\mathbf{k} = -2i - 4j - 5k = \langle -2, -4, -5 \rangle$$

Next find the magnitude:

$$\|\mathbf{v} \times \mathbf{w}\| = \|\langle -2, -4, -5 \rangle\| = \sqrt{(-2)^2 + (-4)^2 + (-5)^2} = \boxed{\sqrt{45} = 3\sqrt{5}}$$

NOTE: Area is a POSITIVE NUMBER. If you give the answer that asks for area as a negative number or a vector YOU WILL MAKE US SAD

c) (Volume of a Parallelipiped)

Find the volume of a parallelipiped spanned by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

Solution:

The VOLUME of the PARALLELIPIPED spanned by the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is given by THE ABSOLUTE VALUE OF THE TRIPLE PRODUCT!

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

We already found the cross product in the last part of the problem. So we just plug and chug into the formula:

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\langle 2, 5, 1 \rangle \cdot \langle -2, -4, -5 \rangle| = |2(-2) + 5(-4) + 1(-5)| = |-29| = \boxed{29}$$

NOTE: Volume is a POSITIVE NUMBER. If you give the answer that asks for volume as a negative number or a vector YOU WILL MAKE US SAD

2 Planes

a) (Normal and Point)

Find the Equation of a plane that is normal to the vector $\mathbf{n} = \langle 3, 1, 8 \rangle$ and passes through the point $P = (1, 2, 3)$

Solution:

Recall from the first midterm, for the equation of a plane, you need a NORMAL VECTOR $\mathbf{n} = \langle A, B, C\rangle$ and a POINT $P = (x_0, y_0, z_0)$. The POINT tells you where the plane is in space, the NORMAL VECTOR tells you which way the plane is facing. Then you just plug it into the equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

You can ALSO use the other two equations:

$$Ax + By + Cz = D = \langle A, B, C\rangle \cdot \langle x_0, y_0, z_0\rangle$$
$$\langle A, B, C\rangle \cdot \langle x, y, z\rangle = D = \langle A, B, C\rangle \cdot \langle x_0, y_0, z_0\rangle$$

You don't have to use all of them. Plugging it in, we get:

$$\boxed{3(x - 1) + 1(y - 2) + 8(z - 3) = 0}$$

You can ALSO use the other two equations:

$$\boxed{3x + y + 8z = 29} = \langle 3, 1, 8\rangle \cdot \langle 1, 2, 3\rangle$$
$$\boxed{\langle 3, 1, 8\rangle \cdot \langle x, y, z\rangle = 29} = \langle 3, 1, 8\rangle \cdot \langle 1, 2, 3\rangle$$

b) (Parallel Planes)

Find the equation of a plane that is parallel to the plane $x+2y+3z = 42$ and passes through the point $P = (10, 5, 3)$

Solution:

If two planes are PARALLEL, then they have THE SAME NORMAL VECTORS, OR THEIR NORMAL VECTORS ARE PARALLEL. the normal vector for the plane that we're parallel to is $\mathbf{n} = \langle 1, 2, 3\rangle$ so we can use this as the normal vector of the plane we're looking for. Now we have anormal vector and a point $P = (10, 5, 3)$. Just plug and chug again:

$$\boxed{(x - 10) + 2(y - 5) + 3(z - 3) = 0}$$

You can ALSO use the other two equations:

$$\boxed{x + 2y + 3z = 29} = \langle 1, 2, 3\rangle \cdot \langle 10, 5, 3\rangle$$
$$\boxed{\langle 1, 2, 3\rangle \cdot \langle x, y, z\rangle = 29} = \langle 1, 2, 3\rangle \cdot \langle 10, 5, 3\rangle$$

c) (Perpendicular Planes)

Find an equation of a plane that is perpendicular to both of the planes found in part a and b.

Solution:

Note that there are an INFINITE number of planes that can be perpendicular to two planes. This problem does not give you a point, nor does it imply there is one. In which case, you can choose any point or any D value. As for the NORMAL VECTOR. Recall that the angle between two planes is the angle between their normal vectors. SO, if two planes are perpendicular, then their normal vectors are perpendicular. And if a plane is perpendicular to TWO planes, then it must be perpendicular to BOTH of their normal vectors. How do you find a vector that is perpendicular to TWO different vectors? TAKE THE CROSS PRODUCT!

So we have the equation fo rthe two planes from parts a) and b):

$$\begin{aligned} x + 2y + 3z &= 29 \\ 3x + y + 8z &= 29 \end{aligned}$$

So we cross the two normal vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle 3, 1, 8 \rangle$:

$$\begin{aligned} \mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 1 & 8 \end{vmatrix} &= i \begin{vmatrix} 2 & 3 \\ 1 & 8 \end{vmatrix} - j \begin{vmatrix} 1 & 3 \\ 3 & 8 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \\ &= (16 - 3)\mathbf{i} - (8 - 9)\mathbf{j} + (1 - 6)\mathbf{k} = 13i + j - 5k = \langle 13, 1, -5 \rangle \end{aligned}$$

So we get $\mathbf{n} = \langle 13, 1, -5 \rangle$. So choose any point. I choose $P = (0, 0, 0)$, because the D value will be 0, and it's easy to compute. Plugging it in, we get:

$$\boxed{13x + y - 5z = 0}$$

You can ALSO use the other two equations:

$$\begin{aligned} \boxed{13x + y - 5z = 0} &= \langle 13, 1, -5 \rangle \cdot \langle 0, 0, 0 \rangle \\ \boxed{\langle 13, 1, -5 \rangle \cdot \langle x, y, z \rangle = 0} &= \langle 13, 1, -5 \rangle \cdot \langle 0, 0, 0 \rangle \end{aligned}$$

d) (Perpendicular Planes II)

Find an equation of the plane that is perpendicular to both of the planes found in part a and b AND passes through the point $P = (1, 2, 5)$.

Solution:

This time we ARE given a point. And we have a normal vector. SO, just plug it into the equations as normal, with $\mathbf{n} = \langle 13, 1, -5 \rangle$
Plugging it in, we get:

$$\boxed{13(x-1)+1(y-2)-5(z-5)=0}$$

You can ALSO use the other two equations:

$$\boxed{13x+y-5z=-10} = \langle 13,1,-5\rangle \cdot \langle 1,2,5\rangle$$
$$\boxed{\langle 13,1,-5\rangle \cdot \langle x,y,z\rangle = -10} = \langle 13,1,-5\rangle \cdot \langle 1,2,5\rangle$$

3 Arc Length and Vector Valued Functions

a) (Eggers' Problem)

Suppose that for some parametrized curve $r(t)$, we have $r'(t)\cdot r''(t)=0$. If $r'(0)=\langle 2,2,1\rangle$, find

$$\int_0^6 \|r'(t)\|dt$$

Solution:

Now this one is hard because we don't have a function $\mathbf{r}(t)$. what we do have is that fact above: $\mathbf{r}'(t)\cdot\mathbf{r}''(t)=0$. It's hard to actualy come up with a solution unless you REALLY think about this. BUT. Remember that $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}$. SO, $\|r'(t)\|^2=\mathbf{r}'(t)\cdot\mathbf{r}'(t)$. Then take the derivative using the product rule:

$$(\mathbf{r}'(t)\cdot\mathbf{r}'(t))' = \mathbf{r}'(t)\cdot\mathbf{r}''(t)+\mathbf{r}''(t)\cdot\mathbf{r}'(t)=0+0=0$$

But since the derivative of $\|\mathbf{r}'(t)\|^2$ is 0, that means $\|\mathbf{r}'(t)\|^2$ is CONSTANT, so $\|\mathbf{r}'(t)\|$ is ALSO constant. Which means $\|\mathbf{r}'(t)\|=\|\mathbf{r}'(0)\|=\sqrt{2^2+2^2+1^2}=3$. Now we just plug and chug.

$$\int_0^6 \|r'(t)\|dt = \int_0^6 3dt = 3t\Big|_{t=0}^{6} = 3(6)-3(0) = \boxed{18}$$

b) (Easier Arc Length Problem)

Compute the arc length of the curve over the given interval

$$\mathbf{r}(t)=\langle 2t,\ln t,t^2\rangle \qquad 1\le t\le 4$$

Solution:

So we need the formula for the arc length, which is given by:

$$\int_a^b \|\mathbf{r}'(t)\|dt$$

So first we calculate $\mathbf{r}'(t)$, then we find the magnitude.

$$\mathbf{r}'(t) = \langle 2, \tfrac{1}{t}, 2t \rangle \implies \|\mathbf{r}'(t)\| = \sqrt{2^2 + (\tfrac{1}{t})^2 + (2t)^2} = \sqrt{4 + \tfrac{1}{t^2} + 4t^2}$$

REMEMBER: The MAGNITUDE of the velocity is going to be a SINGLE VARIABLE FUNCTION THAT IS NOT A VECTOR VALUED FUNCTION. Now from here, we need to simplify. This LOOKS like a nightmare to integrate, but DON'T GIVE UP! Take a look and see what you can do One way to deal with this is to place everything over a common denominator, THEN see if anything cancels.

$$\|\mathbf{r}'(t)\| = \sqrt{4 + \tfrac{1}{t^2} + 4t^2} = \sqrt{\frac{4t^2}{t^2} + \frac{1}{t^2} + \frac{4t^4}{t^2}} = \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}}$$

Under that square root, the numerator is a PERFECT SQUARE!! Basically, $4t^4 + 4t^2 + 1 = (2t^2 + 1)^2$.So we have our function:

$$\|\mathbf{r}'(t)\| = \sqrt{\frac{(2t^2 + 1)^2}{t^2}} = \frac{2t^2 + 1}{t} = 2t + \frac{1}{t}$$

Then we plug it all in TO GET THE ARC LENGTH:

$$\int_a^b \|\mathbf{r}'(t)\| dt = \int_1^4 2t + \frac{1}{t} dt = t^2 + \ln t \Big|_{t=1}^{4} = 4^2 + \ln(4) - 1^2 - \ln(1) = \boxed{15 + \ln(4)}$$

4 Directional Derivatives and Gradient

Let

$$f(x, y) = ye^{y^2 - x}$$

And let $P = (1, 1)$

a) (Gradient)

Find $\|\nabla f_P\|$, the magnitude of the gradient of f at P

Solution:

First we find the gradient:

$$\nabla f(x, y) = \langle -ye^{y^2 - x}, 2y^2 e^{y^2 - x} + e^{y^2 - x} \rangle$$

Then we plug in $P = (1, 1)$

$$\nabla f(1, 1) = \langle -1e^{1^2 - 1}, 2(1)^2 e^{1^2 - 1} + e^{1^2 - 1} \rangle = \langle -1e^0, 2e^0 + e^0 \rangle = \langle -1, 3 \rangle$$

Then we find the magnitude at $P = (1, 1)$

$$\|\nabla f(1, 1)\| = \|\langle -1, 3 \rangle\| = \sqrt{(-1)^2 + 3^2} = \sqrt{1 + 9} = \boxed{\sqrt{10}}$$

b) (Derivative with respect to vector v)

Let $\mathbf{v} = \langle 1, 2 \rangle$. Find the derivative $D_{\mathbf{v}} f(P)$

Solution:

IMPORTANT IMPORTANT IMPORTANT: THIS IS NOT THE SAME THING AS THE DIRECTIONAL DERIVATIVE. YOU DO NOT NORMALIZE THE VECTOR. THIS IS THE DERIVATIVE WITH RESPECT TO **v**!!!!

The derivative with respect to vector **v** at point P is given by the following formula:

$$D_{\mathbf{v}} f(P) = \nabla f_P \cdot \mathbf{v}$$

So we plug in vectors (we got the first part from part (a), the second part is given)

$$D_{\mathbf{v}} f(P) = \nabla f_P \cdot \mathbf{v} = \langle -1, 3 \rangle \cdot \langle 1, 2 \rangle = -1(1) + 3(2) = -1 + 6 = \boxed{5}$$

c) (Angles and Gradients)

Find the rate of change of f in the direction of a vector making a 45° angle with ∇f_P.

Solution:

This uses the alternate formula for directional derivative (THIS IS A DIRECTIONAL DERIVATIVE), which is (with **u** being a UNIT VECTOR):

$$D_{\mathbf{u}} f(P) = \|\nabla f_P\| \cos\theta$$

So we plug everything into the formula:

$$D_{\mathbf{u}} f(P) = \|\nabla f_P\| \cos\theta = \sqrt{10}\cos(45) = \sqrt{10}(\tfrac{\sqrt{2}}{2}) = \boxed{\sqrt{5}}$$

5 Implicit Differentiation

a) (Implicit Partial)

Calculate the partial derivative $\frac{\partial z}{\partial y}$ using implicit differentiation:

$$e^{xy} + \sin(xz) + y = 0$$

Solution

NOTE: When taking partial derivatives implicitly, YOU HOLD THE THIRD VARIABLE CONSTANT, just like in the explicit case. Since you're finding $\frac{\partial z}{\partial y}$, x IS HELD CONSTANT.

So, we use the chain rule:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0$$

Since x is held CONSTANT here, $\frac{\partial x}{\partial y} = 0$ (the dreivative of a constant is 0). The derivative of a variable with respect to itself is 1, so $\frac{\partial y}{\partial y} = 1$. The equation above becomes:

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0$$

Then we solve for $\frac{\partial z}{\partial y}$, and we get:

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

So we find F_y and F_z. Keep in mind that $F = e^{xy} + \sin(xz) + y$.

$$F_y = xe^{xy} + 1 \qquad F_z = x\cos(xz)$$

Then plug and chug into the formula!

$$\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xe^{xy} + 1}{x\cos(xz)}}$$

NOTE: In terms of notation, you could also write z_y instead of $\frac{\partial z}{\partial y}$.

For those of y'all who like their formila sheets, here's a couple:

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \quad \Bigg| \quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

b) (Implicit Tangent Plane)

Find the equation of the tangent plane at point $P = (1, 0, \pi)$.

Solution:

REMEMBER THE PROBLEM FROM MIDTERM 2!! The equation of the tangent plane for $P = (a, b, c)$ is given by:

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

From part a), we have:

$$F_y(x, y, z) = xe^{xy} + 1 \qquad F_z(x, y, z) = x\cos(xz)$$

Then we find $F_x(x, y, z)$

$$F_x(x, y, z) = ye^{xy} + z\cos(xz)$$

So then we find the values of the partial derivatives at the point:

$$F_x(1,0,\pi) = 0e^{1(0)}+\pi\cos(1(\pi)) = -\pi \quad F_y(1,0,\pi) = 1e^{1(0)}+1 = 2 \quad F_z(1,0,\pi) = 1\cos(1\pi) = -1$$

Then plug everything in:

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = \boxed{-\pi(x-1) + 2y - 1(z-\pi) = 0}$$

c) (Linear Approximation)

Find the formula for the linear approximation $L(x,y)$ using the point $P = (1,0,\pi)$ for the implicit surface in part a)

Solution:

The linearization $L(x,y)$ has the SAME EQUATION AS THE TANGENT PLANE WHEN YOU SOLVE FOR z.

$$-\pi(x-1)+2y-1(z-\pi) = -\pi(x-1)+2y-z+\pi = 0 \implies z = \boxed{L(x,y) = -\pi(x-1)+2y+\pi}$$

6 Global Extremes and Optimization

a) (Triangle Domain)

Determine the Global Extremes of the function with the following domain (Hey, remember on the second midterm E-mails how I said he won't ask about Domain on the midterm? Well he might ask about it on the final.)

$$f(x,y) = x^3 + x^2y + 2y^2, \quad x, y \geq 0, \quad x + y \leq 1$$

Solution:

NOTE: The DOMAIN is the triangle enclosed by the three lines.

Step 1: Find the Critical Points

$$f_x(x,y) = 3x^2 + 2xy = x(3x+2y), \quad f_y(x,y) = x^2 + 4y$$

Set the $f_x = 0 = x(3x+2y)$, and we get that $x = 0$ or $y = -\frac{3}{2}x$. When $x = 0$, we have $f_y = x^2 + 4y = 0^2 + 4y = 4y = 0$. So $y = 0$ when $x = 0$, and $(0,0)$ is a critical point.When $y = -\frac{3}{2}x$, we have $f_y = x^2 + 4y = x^2 - 4\frac{3}{2}x = x^2 - 6x = x(x-6) = 0$. So $x = 0$ when $y = -\frac{3}{2}(0) = 0$, and $(0,0)$ is a critical point, and $x = 6$ when $y = -\frac{3}{2}(6) = -9$, and $(6,-9)$ is a critical point. Since $y \geq 0$ on the domain, and $-9 < 0$, we have that $(6,-9)$ is NOT in the domain and you dont have to check it. $(0,0)$, however, is in the domain.

Step 2: Check the Critical Points with the Second Derivative Test

So first we calculate the second order partials.

$$f_{xx}(x,y) = 6x + 2y \qquad f_{xx}(0,0) = 6(0) + 2(0) = 0$$
$$f_{yy}(x,y) = 4 \qquad f_{yy}(0,0) = 4$$
$$f_{xy}(x,y) = 2x \qquad f_{xy}(0,0) = 2(0)$$

Next we find the Discriminant:

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}^2(0,0) = 4(0) - 0^2 = 0$$

Oops, the second derivative tests tells us nothing in this case.

Step 2.5: Find the function values at the critical points

Okay, but we should still find the function values at the critical points. So

$$f(0,0) = 0^3 + (0)^2(0) + 2(0)^2 = 0$$

Remember this for later

Step 3: Check any endpoints of the boundary

I would do this even if the ook doesn't explicitly say to do this. That's becase when you check the boundary itself, the endpoints could be global extremes when you restrict the function to the boundary. If your boundary is a square or triangle or some other shape with corners, the "endpoints" are the corners. In this case, the corners are $(1,0)$, $(0,1)$, and $(0,0)$. So we find the function values at those points. We already found $f(0,0)$, so try to find $f(1,0)$ and $f(0,1)$.

$$f(1,0) = 1^3 + (1)^2(0) + 2(0)^2 = 1 \quad f(0,1) = 0^3 + (0)^2(1) + 2(1)^2 = 2$$

Remember this for later

Step 4: Check the Boundary and its critical points

So asically what this means is take the functions of the boundary, plug it into the original equation, and find the critical points along the boundary. So we have 3 boundary functions: $x = 0$, $y = 0$, and $x + y = 1$, or $y = -x + 1$.

First set $x = 0$. Then

$$f(x,y) = 0^3 + (0)^2(y) + 2(y)^2 = 2y^2 \quad f_x(x,y) = 0 \quad f_y(x,y) = 4y$$

But this only has critical points along the line $y = 0$. But since this is along the border $x = 0$, we have that the only critical point along this border is $(0,0)$, and we already checked that.

Next set $y = 0$. Then

$$f(x, y) = x^3 + x^2(0) + 2(0)^2 = x^3 \quad f_x(x, y) = 3x^2 \quad f_y(x, y) = 0$$

But this only has critical points when $x = 0$. But since this is along the border $y = 0$, we have that the only critical point along this border is $(0, 0)$, and we already checked that.

Next set $y = -x + 1$. Then

$$f(x, y) = x^3+x^2(-x+1)+2(-x+1)^2 = x^3-x^3+x^2+2x^2-4x+2 = 3x^2-4x+2 \quad f_x(x, y) = 6x-4 \quad f_y(x, y) = 0$$

Since along this border, f_y is ALWAYS 0, we only need to set f_x to zero to find the critical points along this border. So we have $f_x = 6x - 4 = 0$, or that $x = \frac{2}{3}$. Along the border, $y = -x + 1 = -\frac{2}{3} + 1 = \frac{1}{3}$. FINALLY, we check the value of the function at this point.

$$f(\tfrac{2}{3}, \tfrac{1}{3}) = \tfrac{2}{3}^3 + (\tfrac{2}{3})^2(\tfrac{1}{3}) + 2(\tfrac{1}{3})^2 = \tfrac{8}{27} + \tfrac{4}{27} + \tfrac{2}{9} = \tfrac{12}{27} + \tfrac{6}{27} = \tfrac{18}{27} = \boxed{\tfrac{2}{3}}$$

Remember this.

Step 5: Figure out the Global Min and Max

We have:

$$f(0, 0) = 0, \quad f(1, 0) = 1, \quad f(0, 1) = 2, \quad f(\tfrac{2}{3}, \tfrac{1}{3}) = \tfrac{2}{3}$$

So how do we find the global max and min? Basically just check all the values above; the highest value above would be the global max and the lowest value above would be the global min.

$$\boxed{\text{global min is } f(0, 0) = 0 \text{ at point } P = (0, 0), \quad \text{global max is } f(0, 1) = 2 \text{ at point } P = (0, 1)}$$

b) (Optimizing with a Constraint)

The cylinder $x^2 + y^2 = 1$ intersects the plane $x + z = 1$ in an ellipse. Find the point on the ellipse that is farthest from the origin.

Solution:

STEP 1: WHAT DO I DO FIRST? Figuring out the equations

PROTIP: IF THE PROBLEM ASKS FOR THE "SHORTEST", "LONGEST", "BIGGEST", "SMALLEST", "MAXIMUM", "MINIMUM", "SEXIEST", ETC, THEN YOU CAN MAKE IT INTO AN OPTIMIZATION PROBLEM. In this case, you can make this problem a Lagrange multiplier problem.

You want to find the LONGEST DISTANCE from the intersection of these TWO SURFACES. You're trying to optimize distance from the origin! Problem is, this is a Lagrange Multiplier question with TWO constraints. So you have TWO multipliers, λ and μ.
The Lagrange Equation becomes:

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

In this case, we try to optimize the distance function (since this is a question that is optimizing distance from the origin, $(a, b, c) = (0, 0, 0)$):

$$f(x, y, z) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = \sqrt{x^2 + y^2 + z^2}$$

PROTIP: Whenever you try to optimize the distance function, you can optimize the function SQUARED. It makes your life easier, trust me on this one. Why? Because partial derivatives are easier to take, and Lagrange equations are easier to solve. If d^2 is minimum/maximum on the constraint, then d must also be minimum/maximum.

$$f(x, y, z) = x^2 + y^2 + z^2$$

You also have your constraint equations:

$$g(x, y, z) = x^2 + y^2 - 1 = 0 \quad \Big| \quad h(x, y, z) = x + z - 1 = 0$$

Now that we have all our equations, we find the gradients:

Step 2: Find the Gradients of the functions, and then find the Lagrange Equation

$$\nabla f = \langle 2x, 2y, 2z \rangle \quad \Big| \quad \nabla g = \langle 2x, 2y, 0 \rangle \quad \Big| \quad \nabla h = \langle 1, 0, 1 \rangle$$

So now we have Lagrange Equations:

$$\langle 2x, 2y, 2z \rangle = \lambda \langle 2x, 2y, 0 \rangle + \mu \langle 1, 0, 1 \rangle$$

Step 3: Solve Lagrange Equations:

Okay, so here's the hard part. We try to solve the equations.

$$\langle 2x, 2y, 2z \rangle = \langle 2\lambda x + \mu, 2\lambda y, \mu \rangle$$

So first, take a look at the second components. $2y = 2\lambda y$. Subtract $2y$ from both sides and we get $2\lambda y - 2y = 2y(\lambda - 1) = 0$. So either $y = 0$ or $\lambda = 1$.

Case 1: If $\lambda = 1$, then from the first components, $2x = 2\lambda x + \mu = 2x + \mu$. This means that $\mu = 0$. But since, from the third components, $2z = \mu$, $z = 0$. We have to follow the constraint $x + z = 1$, so that means that $x = 1$. We also have to follow the constraint

$x^2 + y^2 = 1$, so that $1 + y^2 = 1$ and $y = 0$.

WE GET THE CRITICAL POINT: $(1, 0, 0)$.
Case 2: If $y = 0$, then we can plug y into the constraint equation $x^2 + y^2 = 1$ to get: $x^2 + 0^2 = x^2 = 1$. So $x = \pm 1$. If $x = 1$, then using the other constraint equation, $x + z = 1 + z = 1$, and $z = 0$. If $x = 1$, then using the same constraint equation, $x + z = -1 + z = 1$, and $z = 2$.

WE GET THE CRITICAL POINTS: $(1, 0, 0)$ AND $(-1, 0, 2)$. Then we plug them into the original equation f.

Step 4: Figure out which is the maximum

To figure out which point is the maximum and which is the minimum, just plug in the point into the original equation. Since the domain and function is bounded, there is at least one solution.
REMEMBER: We're using

$$f(x, y, z) = x^2 + y^2 + z^2$$

Plugging it in, we get:

$$f(1, 0, 0) = 1^2 + 0^2 + 0^2 = 1 \quad \Big| \quad f(-1, 0, 2) = (-1)^2 + 0^2 + 2^2 = 5$$

So the point that is FARTHEST from the origin is the one that returned the MAXIMUM value on the objective function f, which in this case is $\boxed{P = (-1, 0, 2)}$.

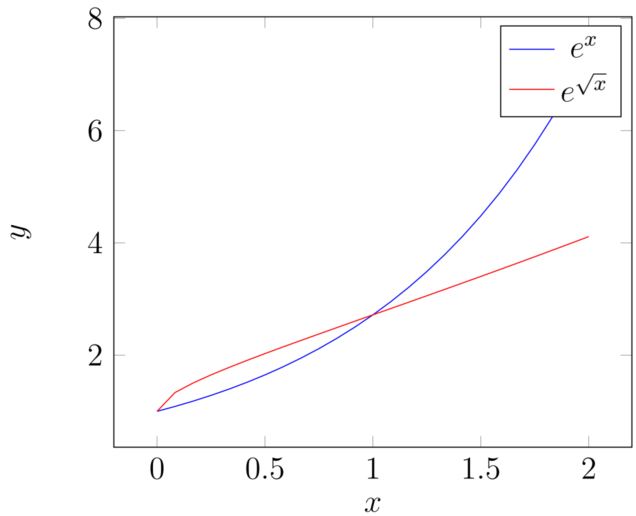
7 Double Integrals

Compute the integral of $f(x, y) = (\ln y)^{-1}$ Over the domain $\mathcal{D}$ bounded by $y = e^x$ and $y = e^{\sqrt{x}}$

Solution:

Step 1: Determine the bounds

For any doble integral problems it's a good idea to graph the domain.



So the top graph is $y = e^{\sqrt{x}}$ and the bottom graph is $y = e^x$. The two graphs intersect at $(0,1)$ and $(1,e)$. So, we can try to integrate the region with $e^x \le y \le e^{\sqrt{x}}$ and $0 \le x \le 1$.

$$\int_{x=0}^{1} \int_{y=e^x}^{e^{\sqrt{x}}} \frac{1}{\ln y} dy dx$$

PROBLEM! Integrating this with respect to y firt is damn near impossible. So you want to make it so you can integrate with respect to x first. But this means that you FIRST integrate from the left function to the right function (both functions of x in terms of y) then with constant bounds for y. So first, you want to put the functions in terms of y, so we can integrate with respect to x first.

$$y = e^x \implies x = \ln y \quad \Big| \quad y = e^{\sqrt{x}} \implies \sqrt{x} = \ln y \implies x = (\ln y)^2$$

So from above, we see that $x = \ln y$ is the right function and $x = (\ln y)^2$ is the left function. Next, we see where y ranges. But since the points of intersection are $(0,1)$ and $(1,e)$, y ranges from 1 to e.

So our bounds for x are: $(\ln y)^2 \le x \le \ln y$ and $1 \le y \le e$. Now we can take the integral.

$$\int_{y=1}^{e} \int_{x=(\ln y)^2}^{\ln x} \frac{1}{\ln y} dx dy = \int_{y=1}^{e} \frac{x}{\ln y}\Big|_{x=(\ln y)^2}^{\ln x} dy = \int_{y=1}^{e} \frac{\ln y}{\ln y} - \frac{(\ln y)^2}{\ln y} dy$$
$$= \int_{y=1}^{e} 1 - \ln y dy = \int_{y=1}^{e} dy - \int_{y=1}^{e} \ln y dy$$

The second integral, you'll have to use integration by parts. For those of you who don't remember, here's the formula!

$$\int u dv = uv - \int v du$$

You want to set something to u and something to dv, and then find du and v (the derivative of u and the integral of dv, respectively), and plug in and solve. A handy way of figuring out which function to choose for u is LIPET:

1. L - Logarithmic functions (ln, log, etc)
2. I - Inverse Trig Functions (arctan, arcsin, arccos, etc)
3. P - Polynomials (x^2, $x^3 + 2x + 1$, etc)
4. E - Exponentials (e^2x, 2^x, etc)
5. T - Trig Functions (sin, cos, tan, etc)

In this case we have $\ln y$ which is right at the top of the list, so we can make that our u. That means dv is everything else, or dy. So we have that:

$$u = \ln y, \quad dv = dy \implies du = \frac{dy}{y}, \quad v = y$$
$$\implies \int \ln y dy = y \ln y - \int y \frac{dy}{y} = y \ln y - \int dy = y \ln y - y$$

So solving for the actual integral, we get:

$$\int_{y=1}^{e} \int_{x=(\ln y)^2}^{\ln x} \frac{1}{\ln y} dx dy = \int_{y=1}^{e} dy - \int_{y=1}^{e} \ln y dy$$
$$= \left(y - (y \ln y - y) \right) \Big|_{y=1}^{e} = \left(2y - y \ln y \right) \Big|_{y=1}^{e}$$
$$= 2e - e \ln e - 2(1) + 1 \ln 1 = \boxed{e - 2}$$

8 Triple Integrals

a) (Volume)

Find the volume of the solid in the octant $x \geq 0$, $y \geq 0$, $z \geq 0$ bounded by $x + y + z = 1$ and $x + y + 2z = 1$.

Solution:

So we have two planes that are not parallel (we know they aren't parallel because they have different normal vectors). For any triple integral problem, the first thing you want to do is calculate the bounds. A good idea to start is to calculate the bounds in terms of z. So we solve for z in the bounds above.

Step1: Calculate the bounds

For the first bound, we have $z = 1 - x - y$ and for the second bound, we have $z = \frac{1-x-y}{2}$. For these bounds, (since $x \geq 0, y \geq 0, z \geq 0$) we have that $1 - x - y \geq \frac{1-x-y}{2}$, so for the first set of bounds: $\frac{1-x-y}{2} \leq z \leq 1 - x - y$.

For the second set of bounds, we need to know the domain of integration. For that, we need to know where the surfaces intersect. The intersection of the surfaces form the rest of the simple domain. So setthe z equal to each other: $1 - x - y = \frac{1-x-y}{2}$. And then solve for y. You get $y = -x + 1$. We also have $y \geq 0$, so our bounds for y are $0 \leq y \leq -x + 1$.

Graphing the domain, you get that the simple domain is the right triangle with corners at the origin, $(1, 0)$, and $(0, 1)$. That means x ranges between 0 and 1, and our final set of bounds are $0 \leq x \leq 1$.

Step 2: Put the integral together and Integrate!

So now that we have our bounds, we just use the formula for volume:

$$V = \iiint_{\mathcal{W}} 1dV = \int_{x=0}^{1} \int_{y=0}^{-x+1} \int_{z=\frac{1-x-y}{2}}^{1-x-y} dzdydx$$

Then we integrate to iterated integral; Integrate with respect to z, then y, then x:

$$\begin{aligned}
V = \iiint_{\mathcal{W}} 1dV &= \int_{x=0}^{1} \int_{y=0}^{-x+1} \int_{z=\frac{1-x-y}{2}}^{1-x-y} dzdydx \\
&= \int_{x=0}^{1} \int_{y=0}^{-x+1} z\Big|_{z=\frac{1-x-y}{2}}^{1-x-y} dV = \int_{x=0}^{1} \int_{y=0}^{-x+1} 1 - x - y - \frac{1-x-y}{2} dydx \\
&= \int_{x=0}^{1} \int_{y=0}^{-x+1} \frac{1-x-y}{2} dydx \\
&= \int_{x=0}^{1} \frac{y}{2} - \frac{xy}{2} - \frac{y^2}{4}\Big|_{y=0}^{-x+1} dx = \frac{1}{2} \int_{x=0}^{1} (1-x) - x(1-x) - \frac{(1-x)^2}{2} dx \\
&= \frac{1}{2} \int_{x=0}^{1} \frac{1}{2} - x + \frac{x^2}{2} dx = \frac{1}{2}\Big(\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6}\Big)\Big|_{x=0}^{1} = \frac{1}{2}\Big(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\Big) = \boxed{\frac{1}{12}}
\end{aligned}$$

b) (Cylindrical)

Express the following triple integral in cylindrical coordinates, then evaluate.

$$\int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{0}^{x^2+y^2} dzdydx$$

Solution:

Step 1: Change bounds to cylindrical coordinates

So this is a VERY important concept: How do you go about converting one integral in one coordinate system into an integral in another cordinate system? In cylindrical coordinates, z is still the same, but x and y are converted to polar coordinates. For those of you who need review, you make these changes to convert x and y to polar coordinates:

$$x = r\cos\theta \quad \Big| \quad y = r\sin\theta \quad \Big| \quad x^2 + y^2 = r^2$$

Also, keep in mind that for the integrals with respect to x and y, the bound are $0 \leq y \leq \sqrt{1-x^2}$, and $-1 \leq x \leq 1$. But if you graph this, this should be the top half of the unit circle. In polar coordinates, this means that $0 \leq r \leq 1$ (because it is the unit circle) and $0 \leq \theta \leq \pi$ (since we're on the top half of the unit circle). We also have that the upperbound for the z integral is $x^2 + y^2$, which we need to convert to polar coordinates. From above, this means $0 \leq z \leq r^2$. Finally, PLEASE KEEP THIS IN MIND:

Step 2: Change the volume element from Euclidean to cylindrical

$$dV = dzdydx = rdzdrd\theta$$

THIS IS THE CHANGE OF VARIABLES FOR THE VOLUME ELEMENT. REMEMBER THAT YOU ADD AN "r" THERE WHEN SWITCHING FROM EUCLIDEAN TO CYLINDRICAL/POLAR COORDINATES.

Step 3: Plug in elements and evaluate

Now we plug everything in:

$$\int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{0}^{x^2+y^2} dzdydx = \int_{\theta=0}^{\pi} \int_{r=0}^{1} \int_{0}^{r^2} rdzdrd\theta$$

And then we integrate!

$$\begin{aligned}
\int_{\theta=0}^{\pi} \int_{r=0}^{1} \int_{0}^{r^2} rdzdrd\theta &= \int_{\theta=0}^{\pi} \int_{r=0}^{1} rz\Big|_{0}^{r^2} drd\theta \\
&= \int_{\theta=0}^{\pi} \int_{r=0}^{1} r(r^2-0)drd\theta = \int_{\theta=0}^{\pi} \int_{r=0}^{1} r^3 drd\theta \\
&= \int_{\theta=0}^{\pi} \frac{r^4}{4}\Big|_{r=0}^{1} d\theta = \int_{\theta=0}^{\pi} \frac{1^4-0^4}{4} d\theta = \int_{\theta=0}^{\pi} \frac{1}{4} d\theta \\
&= \frac{\theta}{4}\Big|_{\theta=0}^{\pi} = \frac{\pi - 0}{4} = \boxed{\frac{\pi}{4}}
\end{aligned}$$