Math 20C Practice Midterm 1 Solutions

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January 24th, 2015

1

Let $\mathbf{u} = \langle 1, 2, -2\rangle$ and $\mathbf{v} = \langle 1, -3, 2\rangle$ and $\mathbf{w} = \langle 11, -3, -2\rangle$

a)

Compute $\mathbf{u} \times \mathbf{v}$

Solution:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 1 & -3 & 2 \end{vmatrix} = i\begin{vmatrix} 2 & -2 \\ -3 & 2 \end{vmatrix} - j\begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} + k\begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix}$$
$$= (4-6)\mathbf{i} - (2-(-2))\mathbf{j} + (-3-2)\mathbf{k} = \boxed{-2i - 4j - 5k = \langle -2, -4, -5\rangle}$$

b)

Find the area of the paralellogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

Solution:

The area of a parallelogram spaned by two vectors is given by the magnitude of their cross product.

$A = ||\mathbf{u} \times \mathbf{v}|| = \sqrt{(-2)^2 + (-4)^2 + (-5)^2} = \sqrt{4 + 16 + 25} = \sqrt{45} = \boxed{3\sqrt{5}}$

If you forgot how to do the cross product, you can also use: $A = ||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||||\mathbf{v}|| \sin(\theta)$, where θ is the angle between the two vectors. That version of the solution is, however, much more difficult to compute, and requires understanding of how the inverse trig functions work.

c)

Express $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Solution:

What does this mean? It means you can write: $\mathbf{w} = r\mathbf{u} + s\mathbf{v}$ for some constant r and s.

In other words:
$\langle 11, -3, -2\rangle = r\langle 1, 2, -2\rangle + s\langle 1, -3, 2\rangle = \langle r + s, 2r - 3s, -2r + 2s\rangle$
This leads to the following system of equations:

$$\begin{aligned} r + s &= 11 \\ 2r - 3s &= -3 \\ -2r + 2s &= -2 \end{aligned}$$

There are a few ways to solve this system. I solved for s in the third equation:

$$-2r + 2s = -2 \implies 2s = 2r - 2 \implies s = r - 1$$

Then plugged it into the first equation,

$$r+s = 11 \implies r+(r-1) = 11 \implies 2r-1 = 11 \implies 2r = 12 \implies r = 6 \implies s = 6-1 = 5$$

Then I would CHECK to see if this works for all of the equations, in other words check that this solution works:

$$\begin{aligned} 6 + 5 &= 11 \checkmark \\ 2(6) - 3(5) &= 12 - 15 = -3 \checkmark \\ -2(6) + 2(5) &= -12 + 10 = -2 \checkmark \end{aligned}$$

Sometimes the solution DOES NOT work, because the vector CANNOT be expressed as a linear combination of the other two vectors. So it's important to check somehow if it works.

But the solution is: $\boxed{\mathbf{w} = 6\mathbf{u} + 5\mathbf{v}}$ or $\boxed{\langle 11, -3, -2\rangle = 6\langle 1, 2, -2\rangle + 5\langle 1, -3, 2\rangle}$

d)

Find $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$

Solution:

Solution 1:

$\mathbf{w} = \langle 11, -3, -2\rangle$ and $\mathbf{u} \times \mathbf{v} = \langle -2, -4, -5\rangle$

So, $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \langle 11, -3, -2\rangle \cdot \langle -2, -4, -5\rangle = 11(-2) - 3(-4) - 2(-5) = -22 + 12 + 10 = \boxed{0}$

Solution 2:

You don't actually need to compute a dot product for this one, but you can. But since $\mathbf{w} = 6\mathbf{u} + 5\mathbf{v}$, you can also do this:

$$\begin{aligned} \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) &= (6\mathbf{u} + 5\mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) \\ &= 6\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) + 5\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) \\ &= 0 + 0 = \boxed{0} \end{aligned}$$

Why is this? Because $\mathbf{u} \times \mathbf{v}$ is vector that is perpendicular to both $\mathbf{u}$ AND $\mathbf{v}$, and when two vectors are perpedicular, their dot product is 0.

e)

Find $\mathbf{e_u}$

Solution:

Remember the fact that $\mathbf{e_u} = \frac{\mathbf{u}}{||\mathbf{u}||}$.

$$\mathbf{e_u} = \frac{\mathbf{u}}{||\mathbf{u}||} = \frac{\langle 1,2,-2\rangle}{\sqrt{1^2+2^2+(-2)^2}} = \frac{\langle 1,2,-2\rangle}{\sqrt{9}} = \frac{\langle 1,2,-2\rangle}{3} = \boxed{\left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right\rangle}$$

f)

Find the projection of $\mathbf{v}$ along $\mathbf{u}$

Solution:

There are two formulas for finding the projection of $\mathbf{v}$ (the vector that you are projecting) along $\mathbf{u}$ (the vector you are projecting onto). They are:

$$\mathbf{v}_{||} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = (\mathbf{v}\cdot\mathbf{e_u})\mathbf{e_u}$$

So either one works. You have the unit vector from part e), so using the second formula:

$$\begin{aligned}
\mathbf{v}_{||} = (\mathbf{v}\cdot\mathbf{e_u})\mathbf{e_u} &= \Big(\langle 1,-3,2\rangle \cdot \langle \tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}\rangle\Big)\langle \tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}\rangle \\
&= \big(1(\tfrac{1}{3}) - 3(\tfrac{2}{3}) + 2(-\tfrac{2}{3})\big)\langle \tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}\rangle \\
&= (\tfrac{1}{3} - 2 - \tfrac{4}{3})\langle \tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}\rangle \\
&= (\tfrac{1}{3} - 2 - \tfrac{4}{3})\langle \tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}\rangle \\
&= (-3)\langle \tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}\rangle \\
&= \boxed{\langle -1,-2,2\rangle}
\end{aligned}$$

g)

Find the point of intersection between the following two lines:

$$\mathbf{r}(t) = \langle 6+t, 12-3t, -12+2t\rangle \qquad \mathbf{s}(t) = \langle 5,-15,10\rangle + t\langle 1,2,-2\rangle$$

Solution:

To find the intersection of two lines, you need to find the point that their parametrizations both reach. So for some value t_1 and t_2, $\mathbf{r}(t)$and $\mathbf{s}(t)$ would reach the same point (NOTE: They might not reach this point at the same time,which is why we need a t_1 and t_2).

$$\mathbf{r}(t_1) = \langle 6+t_1, 12-3t_1, -12+2t_1\rangle \quad \mathbf{s}(t_2) = \langle 5,-15,10\rangle + t_2\langle 1,2,-2\rangle = \langle 5+t_2, -15+2t_2, 10-2t_2\rangle$$

We set the component sequal to each other to obtain the following system:

$$\begin{aligned} 6+t_1 &= 5+t_2 & t_1 &= t_2-1 \\ 12-3t_1 &= -15+2t_2 \implies & -t_1+2t_2 &= 27 \\ -12+2t_1 &= 10-2t_2 & t_1+t_2 &= 11 \end{aligned}$$

Then use the first equation to substitute for t_1 in the third equation, to get:

$$t_2 - 1 + t_2 = 11 \implies 2t_2 = 12 \implies t_2 = 6 \implies t_1 = 6-1 = 5$$

So then we check to see if these values work. We can do this by plugging t_1 into $\mathbf{r}(t)$ and t_2 into $\mathbf{s}(t)$:

$$\mathbf{r}(t_1) = \mathbf{r}(5) = \langle 6+5, 12-3(5), -12+2(5)\rangle = \langle 11,-3,-2\rangle$$
$$\mathbf{s}(t_2) = \mathbf{s}(6) = \langle 5,-15,10\rangle + 6\langle 1,2,-2\rangle = \langle 5,-15,10\rangle + \langle 6,12,-12\rangle = \langle 11,-3,-2\rangle \checkmark$$

So the point of intersection is $\boxed{(11,-3,-2)}$

2

Suppose $\mathbf{u}$ is a unit vector and suppose $\mathbf{v}$ is a vector with $||\mathbf{v}|| = 2$, for which $||\mathbf{u}+\mathbf{v}|| = \frac{3}{2}$. **Find $||4\mathbf{u} - 2\mathbf{v}||$**

Solution:

To solve this problem, you'll want to remember this formula: $||\mathbf{w}||^2 = \mathbf{w}\cdot\mathbf{w}$

You'll also want to remember basic properties of the dot product, like distribution and commutativity (order of the dot product doesn't matter).

$$\begin{aligned} ||\mathbf{u}+\mathbf{v}|| = \frac{3}{2} &\implies ||\mathbf{u}+\mathbf{v}||^2 = \frac{9}{4} \implies (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = ||\mathbf{u}+\mathbf{v}||^2 = \frac{9}{4} \\ (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) &= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} \\ &= \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v} \\ &= ||\mathbf{u}||^2 + 2(\mathbf{u}\cdot\mathbf{v}) + ||\mathbf{v}||^2 \\ &= 1 + 2(\mathbf{u}\cdot\mathbf{v}) + 4 = \frac{9}{4} \\ &\implies 2(\mathbf{u}\cdot\mathbf{v}) = -\frac{11}{4} \implies \mathbf{u}\cdot\mathbf{v} = -\frac{11}{8} \end{aligned}$$

Now we have a value for $\mathbf{u}\cdot\mathbf{v}$. We can use this, and the first formula above to get what we want. Remember that: $||\mathbf{4u} - \mathbf{2v}||^2 = (\mathbf{4u} - \mathbf{2v}) \cdot (\mathbf{4u} - \mathbf{2v})$.

$$\begin{aligned}
(\mathbf{4u} - \mathbf{2v}) \cdot (\mathbf{4u} - \mathbf{2v}) &= \mathbf{4u} \cdot \mathbf{4u} - \mathbf{4u} \cdot \mathbf{2v} - \mathbf{2v} \cdot \mathbf{4u} + \mathbf{2v} \cdot \mathbf{2v} \\
&= 16(\mathbf{u} \cdot \mathbf{u}) - 8(\mathbf{u} \cdot \mathbf{v}) - 8(\mathbf{v} \cdot \mathbf{u}) + 4(\mathbf{v} \cdot \mathbf{v}) \\
&= 16(\mathbf{u} \cdot \mathbf{u}) - 16(\mathbf{u} \cdot \mathbf{v}) + 4(\mathbf{v} \cdot \mathbf{v}) \\
&= 16||\mathbf{u}||^2 - 16(\mathbf{u} \cdot \mathbf{v}) + 4||\mathbf{v}||^2 \\
&= 16 - 16(\mathbf{u} \cdot \mathbf{v}) + 16 = 16 - 16(-\frac{11}{8}) + 16 = 16 + 22 + 16 = 54 \\
&\implies ||\mathbf{4u} - \mathbf{2v}||^2 = 54 \implies ||\mathbf{4u} - \mathbf{2v}|| = \sqrt{54} = \boxed{3\sqrt{6}}
\end{aligned}$$

3

$$P = (5, 15, -10), \quad Q = (20, -5, 10), \quad R = (-1, -1, -1), \quad S = (4, 3, -2), \quad T = (-1, 2, 3)$$

Find a vector parametrization for the line with the given description:

Line that passes through the point on $\overline{PQ}$ lying three fifths ($\frac{3}{5}$) of the way from P to Q, and is perpendicular to the plane that contains points R, S, and T.

Solution:

Okay this one is a bit tricky, because it involves a bunch of moving parts, and it requires you to take two separate parametrizations. But break the problem down a bit. What is it asking for? **The parametrization of a line**. What do you need? **A point (position vector), and a direction vector**. Now you have to find out what those would be.

Step 1 (The point): We know the line "passes through the point on $\overrightarrow{PQ}$ lying three fifths ($\frac{3}{5}$) of the way from P to Q". But how do we find this point? Taking a look at example 7 in section 12.2, you'll see a parametrization that's used to find the midpoint of the line segment:

$$\mathbf{r}(t) = (1 - t)\overrightarrow{P} + t\overrightarrow{Q}$$

The formula in the example uses $t = \frac{1}{2}$ to find the midpoint, or the point halfway between the two points. But to find the point three fifths of the way from P to Q, you use $t = \frac{3}{5}$.

Why? Because when $t = 0$, $\mathbf{r}(0) = \overrightarrow{P}$, and when $t = 1$, $\mathbf{r}(1) = \overrightarrow{Q}$. So in one unit of time, the function traces over the segment from P to Q. But the rate at which it is moving is constant, so at $t = \frac{3}{5}$, $\mathbf{r}(\frac{3}{5})$ is pointing to the point three fifths of the way from P to Q. So to find the point we want, just parametrize the line that contains P and Q, and plug and chug.

$$\begin{aligned}
\mathbf{s}(t) &= (1-t)\overrightarrow{P} + t\overrightarrow{Q} \\
&= (1-t)\langle 5, 15, -10\rangle + t\langle 20, -5, 10\rangle \\
\mathbf{s}(\tfrac{3}{5}) &= (1-\tfrac{3}{5})\langle 5, 15, -10\rangle + \tfrac{3}{5}\langle 20, -5, 10\rangle \\
&= \tfrac{2}{5}\langle 5, 15, -10\rangle + \tfrac{3}{5}\langle 20, -5, 10\rangle \\
&= \langle 2, 6, -4\rangle + \langle 12, -3, 6\rangle \\
&= \langle 14, 3, 2\rangle
\end{aligned}$$

So now we've got a position vector. What about direction?

Step 2 (The direction): So we also know that the line we want is "perpendicular to the plane that contains points R, S, and T". But what does that mean? That means it's going to be parallel to a **normal vector** of the plane (or it's going to be pointing in the same direction as a vector normal to the plane). Knowing how to find that normal vector is very important. Example 3 in section 12.5 goes over this process. First we find two vectors that lie in the plane, and then find their cross product. We can do this by finding $\overrightarrow{RS}$ and $\overrightarrow{RT}$, which lie in the plane:

$$\begin{aligned}
\overrightarrow{RS} &= \langle 4, 3, -2\rangle - \langle -1, -1, -1\rangle = \langle 4-(-1), 3-(-1), -2-(-1)\rangle = \langle 5, 4, -1\rangle \\
\overrightarrow{RT} &= \langle -1, 2, 3\rangle - \langle -1, -1, -1\rangle = \langle -1-(-1), 2-(-1), 3-(-1)\rangle = \langle 0, 3, 4\rangle
\end{aligned}$$

Then we find the cross product $\overrightarrow{RS} \times \overrightarrow{RT}$:

$$\begin{aligned}
\overrightarrow{RS} \times \overrightarrow{RT} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 4 & -1 \\ 0 & 3 & 4 \end{vmatrix} = i\begin{vmatrix} 4 & -1 \\ 3 & 4 \end{vmatrix} - j\begin{vmatrix} 5 & -1 \\ 0 & 4 \end{vmatrix} + k\begin{vmatrix} 5 & 4 \\ 0 & 3 \end{vmatrix} \\
&= (16-(-3))\mathbf{i} - (20-0)\mathbf{j} + (15-0)\mathbf{k} = 19i - 20j + 15k = \langle 19, -20, 15\rangle
\end{aligned}$$

So now we have a direction vector. All we have to do now is put it all together.

Step 3: From step 1, we got that the line passes through the point (14,3,2). From step 2, we got that the line is parallel to the normal vector of the plane that contains R, S, and T, so the direction vector would be $\langle 19, -20, 15\rangle$. You should also know the point-direction form of line parametrization:

$$\mathbf{r}(t) = \overrightarrow{P} + t\overrightarrow{D}$$

Where $\overrightarrow{P}$ is a position vector and $\overrightarrow{D}$ is a direction vector. Just plug everything in:

$$\boxed{\mathbf{r}(t) = \langle 14, 3, 2\rangle + t\langle 19, -20, 15\rangle}$$

4

a)

Find the equation of a plane that contains the line $\mathbf{r}(t) = \langle 3t, t, 2t+1 \rangle$ and is perpendicular to the plane $2x - y + 5z = 9001$. Express your answer in 3 forms (one vector form, and two scalar forms).

Solution:

Okay, another tricky one. But again, ask yourself: What is it asking for? **The equation of a plane**. What do you need? **A point (position vector), and a normal vector**. Now you have to find out what those would be.

Step 1 (The point): How do you find a point? Well you know that the plane contains $\mathbf{r}(t) = \langle 3t, t, 2t+1 \rangle$. So it contains every point on that line. Just get a point on that line. The easiest by default is usually the point at $t = 0$. So point P, or the position vector $\overrightarrow{P}$ is:

$$\overrightarrow{P} = \mathbf{r}(0) = \langle 3(0), 0, 2(0) + 1 \rangle = \langle 0, 0, 1 \rangle$$

Step 2 (The normal vector): This one is harder because you can't just pick three points on the line to find the normal vector, and you aren't given a normal vector. You ARE given that the plane you want is perpendicular to the plane with the equation $2x - y + 5z = 9001$. What does that mean? That means that there is a vector that is on your plane that is perpendicular to the plane that's given. But a vector that is perpendicular to the plane that's given is going to be some scalar times the normal vector to that plane. **In other words, if two planes are perpendicular, they will contain each other's normal vectors**. So a vector normal to the plane $2x - y + 5z = 9001$, like $\langle \mathbf{2}, \mathbf{-1}, \mathbf{5} \rangle$ **is going to lie in the plane you want to find**.

BUT that vector is NOT the vector that is normal to the plane you're looking for. It lies in the plane. **To find the normal vector, you want to find the cross product of two non-parallel vectors in the plane**. Now we have to find the second vector that lies in the plane. Remember that we're given that the plane also contains a line. And **if the plane contains a line, that line's direction vector MUST lie in the plane**.

Aside: Here's how I know that if the plane contains a line, it contains the line's direction vector: At each fixed t, you get a position vector pointing to some point on the line. So $\mathbf{r}(0)$ points to the point on the line where $t = 0$. Since the plane contains the line, it will obviously contain every pointon the line. So $\mathbf{r}(0)$ and $\mathbf{r}(1)$ are position vectors of points on the plane. And to find a vector that lies in a plane, you can subtract the position vectors of any two points on the plane. If $\mathbf{r}(t) = \overrightarrow{P} + t\overrightarrow{D}$:

$$\mathbf{r}(1) - \mathbf{r}(0) = \overrightarrow{P} + (1)\overrightarrow{D} - (\overrightarrow{P} + (0)\overrightarrow{D}) = \overrightarrow{D}$$

In other words, the direction vector $\overrightarrow{D}$ lies on the plane. In this case, the line has direction vector $\langle \mathbf{3}, \mathbf{1}, \mathbf{2} \rangle$. So now that we have two not parallel vectors that lie in the plane, we can find a normal vector by finding the cross product:

$$\langle 2,-1,5\rangle \times \langle 3,1,2\rangle =$$
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 5 \\ 3 & 1 & 2 \end{vmatrix} = i\begin{vmatrix} -1 & 5 \\ 1 & 2 \end{vmatrix} - j\begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix} + k\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix}$$
$$= (-2-5)\mathbf{i} - (4-15)\mathbf{j} + (2-(-3))\mathbf{k} = -7i + 11j + 5k = \langle -7, 11, 5\rangle$$

So now we have the normal vector. The question at this point is: How do we put this all together?

Step 3: The equation of a plane can be written in a few ways. The textbook lists these forms (note that $\mathbf{n} = \langle a,b,c\rangle$ is a normal vector to the plane and $\mathbf{p} = \langle x_0, y_0, z_0\rangle$ is the position vector of some point on the plane and that $d = \mathbf{n}\cdot\mathbf{p} = ax_0 + by_0 + cz_0$):

$$\begin{aligned} \mathbf{n}\cdot\langle x,y,z\rangle &= d \quad \text{(Vector form)} \\ ax+by+cz &= d \quad \text{(Scalar form 1)} \\ a(x-x_0)+b(y-y_0)+c(z-z_0) &= 0 \quad \text{(Scalar form 2)} \end{aligned}$$

We found the point $(0,0,1)$ was on the plane, so we can dot its position vector with the normal vector to get $d = \langle -7,11,5\rangle \cdot \langle 0,0,1\rangle = -7(0)+11(0)+5(1) = 5$. So we plug in all the values to get our three equations:

$$\begin{aligned} \langle -7,11,5\rangle\cdot\langle x,y,z\rangle &= 5 \quad \text{(Vector form)} \\ -7x+11y+5z &= 5 \quad \text{(Scalar form 1)} \\ -7x+11y+5(z-1) &= 0 \quad \text{(Scalar form 2)} \end{aligned}$$

b)

Find $\cos(\theta)$, where θ is the angle between the plane found in part a) and the xz-plane.

Solution:

The angle between two planes is the angle between their normal vectors. The vector that is normal to the plane from part a) was found to be $\mathbf{v} = \langle -7,11,5\rangle$. The easiest vector to use that is normal to the xz-plane is the unit vector $\mathbf{u} = \langle 0,1,0\rangle$. How do we know that this vector is normal to the xz-plane? Because every vector in the xz-plane is of the form $\langle x,0,z\rangle$, and if you dot the two vectors, you'll get 0 every time. Or you'll notice that the equation for the xz-plane is $y = 0$, which means that the normal vector would be $\langle 0,1,0\rangle$.

Now for two vectors $\mathbf{u}$ and $\mathbf{v}$, $\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}$. So calculate:

$$\begin{aligned} &\mathbf{u}\cdot\mathbf{v} = \langle 0,1,0\rangle\cdot\langle -7,11,5\rangle = 0(-7)+1(11)+0(5) = 11 \\ &||\mathbf{u}|| = 1 \\ &||\mathbf{v}|| = \sqrt{(-7)^2+11^2+5^2} = \sqrt{49+121+25} = \sqrt{195} \end{aligned}$$

So $\cos(\theta) = \boxed{\dfrac{11}{\sqrt{195}}}$