12.1 and 12.2: Intro to vectors, line parametrizations

To get the parametrization of a line, you need 2 ingredients:

1. **A point on the line.** The position vector for this point would be: $\mathbf{p} = \langle x_0, y_0, z_0 \rangle$. This tells you where the line is in space. The problem will often say that the line "passes through" a point. That would be a point on the line.

2. **A direction vector.** ($\mathbf{d} = \langle a, b, c \rangle$) This is a non-zero vector that is parallel to the line. The direction vector tells you which way the line is going.

How to get a direction vector:

2 points ON THE LINE OR ON A PARALLEL LINE: Let's say you have $\overrightarrow{\mathbf{P}}$ and $\overrightarrow{\mathbf{Q}}$ on the line or on theparallel line. Then (even if the line is parallel) $\overrightarrow{\mathbf{PQ}} = \overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{P}}$, is a direction vector

1 direction Vector: Sometimes they give you a point and another line, or maybe that the line you are looking for is perpendicular to/parallel to/some other direction to some other line. Then you need to figure out what that means. A couple tips:

1. If the question asks you for a line that is parallel to another line, then they will have the same direction vector

2. If the question asks for a line perpendicular to a plane, then a direction vector can be the plane's normal vector.

3. Do NOT use the 2-point parametrization if one of those points is NOT on the line, or if it is the direction vector.

Getting the parametrization of the line:

Just plug and chug into the equations below (with your point $\mathbf{p} = \langle x_0, y_0, z_0 \rangle$ and your normal vector $\mathbf{d} = \langle a, b, c \rangle$:

$$\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

If you have two points ON THE LINE, like $\mathbf{p} = \langle x_0, y_0, z_0 \rangle$ and $\mathbf{q} = \langle x_1, y_1, z_1 \rangle$, then you can ALSO use the 2-point parametrization

$$\mathbf{r}(t) = t\langle x_0, y_0, z_0 \rangle + (1 - t)\langle x_1, y_1, z_1 \rangle$$

Other important things:

1. Basic properties of a vector, how to find components of a vector, difference between vectors and points

2. MAGNITUDE OF A VECTOR: If you have $\mathbf{v} = \langle a, b, c\rangle = ai + bj + ck$, then the magnitude is $||\mathbf{v}|| = \sqrt{a^2 + b^2 + c^2}$.
3. $\mathbf{v} = ||\mathbf{v}||\mathbf{e_v}$, where $\mathbf{e_v}$ is the unit vector in the same direction as $\mathbf{v}$.
4. Finding intersections of lines
5. $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, $\mathbf{k} = \langle 0, 0, 1\rangle$

12.3: The Dot Product

The **Dot product** $\mathbf{v} \cdot \mathbf{w}$ of TWO VECTORS

$$\mathbf{v} = \langle a, b, c\rangle, \quad \mathbf{w} = \langle d, e, f\rangle$$

is the SCALAR defined by

$$\mathbf{v} \cdot \mathbf{w} = ad + be + cf$$

Other important things:

1. Remember the basic properties of the dot product on pages 678-679 in the textbook (first two pages of section 12.3), like $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$
2. Remember anything in the boxes on pages 680-682 (the next three pages of the section).
3. **ESPECIALLY REMEMBER**: $\mathbf{v} \perp \mathbf{w}$ IF AND ONLY IF $\mathbf{v} \cdot \mathbf{w} = 0$ ($\mathbf{v}$ is perpendicular to $\mathbf{w}$–also known as: orthogonal, makes a 90° angle with $\mathbf{w}$– if and only if their dot product is 0), how to find the projection of a vector along another vector, and how to calculate the angle between two vectors.

12.4: The Cross Product

The **Cross Product** $\mathbf{v} \times \mathbf{w}$ of TWO VECTORS

$$\mathbf{v} = \langle a, b, c\rangle, \quad \mathbf{w} = \langle d, e, f\rangle$$

is the VECTOR defined by:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix} = i \begin{vmatrix} b & c \\ e & f \end{vmatrix} - j \begin{vmatrix} a & c \\ d & f \end{vmatrix} + k \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$
$$= (bf - ce)\mathbf{i} - (af - cd)\mathbf{j} + (ae - bd)\mathbf{k} = \langle bf - ce, cd - af, ae - bd\rangle$$

REMEMBER THIS FORMULA AND HOW TO CALCULATE THE CROSS PRODUCT

Other important things:

1. Remember the basic properties of the cross product on pages 688-690 in the textbook (first three pages of section 12.4)–e.g. the geometric description, how does it work, anti-commutativity (you can't just switch order), etc

2. Remember how to calculate the area of a parallelogram spanned by two vectors, how to calculte a triple product (pg 690-691, Under "Cross products, area, and volume" section 12.4)

12.5: Planes

To get the equation of a plane, you need 2 ingredients:

1. **A point on the plane.** The position vector for this point would be: $\mathbf{p} = \langle x_0, y_0, z_0 \rangle$. This tells you where the plane is in space. The problem will often say that the line "passes through" a point. That would be a point on the plane.

2. **A normal vector to the plane.** ($\mathbf{n} = \langle a, b, c \rangle$) This is a NON-ZERO vector that is perpendicular to every vector that lies on the plane. The normal vector tells you which way the plane is facing.

How to get a normal vector:

3 points THAT ARE NOT ON THE SAME LINE (non-collinear): Either you will be given 3 points, or you have to figure the points out for yourself. You might be given a line and a point, or two lines, or something else. Let's say you have points with position vectors $\overrightarrow{\mathbf{P}}$, $\overrightarrow{\mathbf{Q}}$, and $\overrightarrow{\mathbf{R}}$

2 vectors THAT LIE ON THE PLANE: These are **NOT** the same thing as position vectors that point to points on the plane, so you can't just use $\overrightarrow{\mathbf{P}}$ and $\overrightarrow{\mathbf{Q}}$. Instead, use $\overrightarrow{\mathbf{PQ}} = \overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{P}}$, and $\overrightarrow{\mathbf{PR}} = \overrightarrow{\mathbf{R}} - \overrightarrow{\mathbf{P}}$

1 Normal Vector: Now that you have two vectors that lie on the plane, to find one that is perpendicular to the entire plane, then you just need to cross them. In other words, $\mathbf{n} = \overrightarrow{\mathbf{PQ}} \times \overrightarrow{\mathbf{PR}}$

A couple tips though:

1. If the plane is parallel to a plane $ax + by + cz = d$, then they have the SAME normal vectors $\mathbf{n} = \langle a, b, c \rangle$.

2. If the plane contains two lines $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, then it contains EVERY point on BOTH those lines. So you can find three points that are on those lines by plugging in values of t (do NOT just pick 3 points on the same line).

Getting the equation of the plane:

Just plug and chug into the equations below (with the position vector of your point $\mathbf{p} = \langle x_0, y_0, z_0 \rangle$ and your normal vector $\mathbf{n} = \langle a, b, c \rangle$:

$$\begin{aligned} \mathbf{n} \cdot \langle x, y, z \rangle &= d \quad \text{(Vector form)} \\ ax + by + cz &= d \quad \text{(Scalar form 1)} \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \quad \text{(Scalar form 2)} \\ (d = \mathbf{p} \cdot \mathbf{n} = \langle x_0, y_0, z_0 \rangle \cdot \langle a, b, c \rangle &= ax_0 + by_0 + cz_0) \end{aligned}$$

Other possibly useful things:

1. **Equations for certain planes:** xy-plane: $z = 0$ (normal vector is $\mathbf{n} = \langle 0, 0, 1 \rangle$)
   xz-plane: $y = 0$ (normal vector is $\mathbf{n} = \langle 0, 1, 0 \rangle$)
   yz-plane: $x = 0$ (normal vector is $\mathbf{n} = \langle 1, 0, 0 \rangle$)

2. The angle between two planes is equal to the angle between their normal vectors

Things that came up in the review session that did not originally make it to this handout

1. **SHOW YOUR WORK**. We cannot give you full credit for just the answer, and we can give you some credit if you have some working process. We don't just want to know the answer, because that's easy for us to find. We want to know how you got there. **This goes for your homework as well**.

2. **Know the difference between points (a, b, c) and vector $\langle a, b, c \rangle$ notations.** If the question asks for the answer in the form of a point, use parentheses "( )". If the question asks for the answer in the form of a vector or a parametrization, use angle brackets "$\langle\ \rangle$"

3. **When anything mentions 3 points that are not on the same line or 3 points that are NOT co-linear, it means that NOT ALL 3 of those points are on the same line**. Any 2 of those points will always be on the same line, because you can always draw a line between 2 points in Euclidean space. The idea is that when you connect ALL 3 points, you can form a triangle, and not a line. So in the 2 line plane case (exercise 26 in section 12.5), you can pick two points from one line and one point from the other line that is NOT on the first line to find the normal vector.

4. **The midterm is on sections 12.1-12.5.** It does NOT cover anything in chapter 13, section 12.6, 12.7, or anything from chapter 11. There are 4 questions, at about 20 points each (plus a question 0 that will ask you to follow the directions–please do).

5. **If you know how to do all your homework problems correctly, you should have no problem with the actual midterm.**