A Couple Things to Mention:

1. The midterm will cover sections 13.1-13.3, 13.5, and 14.1-14.5
2. I was instructed to emphasize certain concepts, but the midterm handout probably won't be nearly as helpful as the practice midterm.
3. Things that will NOT be on the midterm: Section 13.4, section 13.6, section 14.6, Graphing, matching 3-D graphs, domain and range problems. He wants you all to have an understanding of graphs in 3-D, and to develop an intuition for visualizing 3-D graphs, but it need not be super detailed.
4. BE CAREFUL–a lot of places in this midterm will likely cover one of the many formulas that were thrown at you in the past several weeks. Particularly, which formulas to use, how to take partial derivatives, and in general, anything from math 20AB/Calc 1/Calc 2/Calc AB/Calc BC, etc, will likely come up here.

Chapter 13: Vector Valued Functions, and Calculus of Vector Valued Functions

THis chapterfocuses on VECTOR-VALUED FUNCTIONS that take in one value (t), and outputs a VECTOR/VECTOR FUNCTION, $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$

Section 13.1: Vector Parametrizations

Two ways given to parametrize a curve:

1. Set one variable to t, and make all other functions into functions of t (NOTE: If you have functions with square roots, then you will likely have TWO parametrizations). See exercise 21 in section 13.1
2. Use trig functions. Example: $x^2 + y^2 = 16$ and $y^2 - x^2 = z - 1$, set $x = 4\cos t$ and $y = 4\sin t$, and solve for z in the second equation

13.2: Calculus of Vector-valued functions

For vector valued functions, a lot of the calculus carries over from single variable calculus to multivariable calculus. You're basically doing the same things, only multiple times.
So let's say:

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, \quad \mathbf{r}_1(t) = \langle x_1(t), y_1(t), z_1(t)\rangle, \quad \mathbf{r}_2(t) = \langle x_2(t), y_2(t), z_2(t)\rangle$$

And $f(t)$ is a SINGLE VARIABLE SCALAR FUNCTION.

Rule	Vector Valued Version
Derivative	$\mathbf{r}'(x) = \lim_{t\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} = \langle x'(t), y'(t), z'(t)\rangle$
Integral	$\int \mathbf{r}(t)dt = \langle \int x(t)dt, \int y(t)dt, \int z(t)dt\rangle$
Product rule	$(f(t)\mathbf{r}(t))' = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$
DOT product rule	$(\mathbf{r_1}(t)\cdot\mathbf{r_2}(t))' = \mathbf{r_1}'(t)\cdot\mathbf{r_2}(t) + \mathbf{r_1}(t)\cdot\mathbf{r_2}'(t)$
CROSS product rule	$(\mathbf{r_1}(t)\times\mathbf{r_2}(t))' = \mathbf{r_1}'(t)\times\mathbf{r_2}(t) + \mathbf{r_1}(t)\times\mathbf{r_2}'(t)$
Chain rule	$(\mathbf{r}(f(t)))' = \mathbf{r}'(f(t))f'(t)$

13.3: Arc Length Parametrization:

The length of a path for $a \le t \le b$ of $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is:

$$\int_a^b ||\mathbf{r}'(t)||dt = \int_a^b \sqrt{x'(t) + y'(t) + z'(t)}dt$$

13.5: Motion in Three Space

Given a parametrization $\mathbf{r}(t)$ that traces out a path, we have that the velocity $\mathbf{v}(t)$, acceleration $\mathbf{a}(t)$, and speed $s(t)$ (speed is a single variable function of t) is defined by:

$$\mathbf{v}(t) = \mathbf{r}'(t) \quad \mathbf{a}(t) = \mathbf{r}''(t) \quad s(t) = ||\mathbf{r}'(t)||$$

Given a parametrization for the ACCELERATION $\mathbf{a}(t)$ that gives you the acceleration at time t, we have that the velocity $\mathbf{v}(t)$, position $\mathbf{r}(t)$, and speed $s(t)$ (speed is a single variable function of t) is defined by:

$$\mathbf{v}(t) = \int \mathbf{a}(t) + \langle c_1, c_2, c_3\rangle \quad \mathbf{r}(t) = \int \mathbf{v}(t) + \langle k_1, k_2, k_3\rangle \quad s(t) = ||\mathbf{v}(t)||$$

Where $c_1, c_2, c_3, k_1, k_2, k_3$ are all constants of integration. You should know what these are and why you need them when integrating.

And we also have Newton's Second Law:

$$\mathbf{F}(t) = m\mathbf{a}(t)$$

Where $\mathbf{F}(t)$ is the force vector valued function, $\mathbf{a}(t)$ is the acceleration vector valued function, and m is the mass. If you're given the force, then you can find the acceleration by just dividing by the mass.

Chapter 14: Multi-variable functions, things in 3D, where everything gets really hard and you might maybe cry

This section focused on multi-variable functions, which take in more than one argument, and output a NUMBER or a VECTOR (depending on what kind of multivariable functions you use)

Chapter 14.1-2: Multi-variable functions, Limits, and Continuity

This handout will NOT cover graphing or domain/range problems, because they will not be on the midterm. Do have an intuition of how things look in 3D.

Three things you can do to find limit:

1) Plug in the variables

If you wantthe limit at point (a, b), and the function is continuous at (a, b), then you just plug in the values of (a, b) into the function. This generally means that you don't get $\frac{0}{0}$, divide by 0, or take the square root of a negative number or other weird things anywhere.

2) Try to make a sandwich/ use squeeze theorem

Basically, you might have a function that you can't plug in the numbers for without getting $\frac{0}{0}$, dividing by 0, or taking the square root of a negative number, or other weird things anywhere. In which case, your next best guess is to make your function easier to deal with. You want to use the **Squeeze Theorem** to trap weird functions into easy, nice functions. If those easy, nice functions approach the same limit, then the weird function, trapped between them, must also approach that limit.

3) Prove the limit does not exist

This one is generally the hardest of the three. You basically want to prove the limit does not exist. In single variable, you could do this by proving that the limit from the left and the limit from the right aren't equal. In multivariable, you just need to prove that the limit isn't the same for any two directions.
NOTE: I go into much more detail about this in the solutions to the practice midterm!!

Chapter 14.3: Partial Derivatives

Partial derivatives are a lot like derivatives in one dimension. The difference is that, in multivariable calculus, you take derivatives in multiple dimensions. Partial derivatives are derivatives in multivariable functions, BUT WITH RESPECT TO ONE VARIABLE. **You hold every other variable constant**. Single variable derivatives are the rate of change in one dimension. **Multi variable partial derivatives are the rates of change with respect to each variable separately.**

Single Variable Derivative (Review):

$$\frac{d}{dx}f(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Multi-Variable Partial Derivatives:

Example equation:

$$g(x, y) = x^2y^3 + \ln x + \sin y$$

IMPORTANT NOTE: When you take the partial derivative with respect to a variable, YOU HOLD ALL OTHER VARIABLE CONSTANT. Do not, Do Not, DO NOT forget that you are treating them as constants, and DO NOT forget that they are there. That is one of the most common ways to lose points on an exam.

Partial Derivative with respect to x:

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

Example equation partial with respect to x (PRETEND y IS CONSTANT):

$g_x(x, y) = 2xy^3 + \dfrac{1}{x}$ Note: y is constant here, so $\sin y$ is constant, and its derivative is 0

Partial Derivative with respect to y:

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Example equation partial with respect to y (PRETEND x IS CONSTANT):

$g_y(x, y) = 3x^2y^2 + \cos y$ Note: x is constant here, so $\ln x$ is constant, and its derivative is 0

Second Order Partial Derivatives:

$$\frac{\partial^2}{\partial y^2} f(x, y) = f_{yy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f(x, y)\right)$$

$$\frac{\partial^2}{\partial x^2} f(x, y) = f_{xx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(x, y)\right)$$

Example equation second order partial with respect to x (PRETEND y IS CONSTANT):

$$g_{xx}(x, y) = \frac{\partial}{\partial x}\left(2xy^3 + \frac{1}{x}\right) = 2y^3 - \frac{1}{x^2}$$

Example equation second order partial with respect to y (PRETEND x IS CONSTANT):

$$g_{yy}(x, y) = \frac{\partial}{\partial y}\left(3x^2y^2 + \cos y\right) = 6x^2y - \sin y$$

Second Order MIXED Partial Derivatives:

$$\frac{\partial^2}{\partial x \partial y} f(x,y) = f_{xy}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right)$$

$$\frac{\partial^2}{\partial y \partial x} f(x,y) = f_{yx}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right)$$

Example equation second order partial with respect to y THEN x :

$$g_{xy}(x,y) = \frac{\partial}{\partial x}\left(3x^2y^2 + \cos y\right) = 6xy^2$$

Example equation second order mixed partial with respect to x THEN y :

$$g_{yx}(x,y) = \frac{\partial}{\partial y}\left(2xy^3 + \frac{1}{x}\right) = 6xy^2$$

Notice that they're equal? That's because of **Clairaut's Theorem**. It basically states **you can take the partial derivaties in any order as long as the partials are continuous.**

Chapter 14.4 Tangent Planes and Linearization (The second tangent plane is from section 14.5)

The idea for this section is that the plane tangent to a point on a nice surface is CLOSE to the surface near the point where it is tangent to. **Linearization of** $z = f(x,y)$ **at** (a,b) is:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

This is exactly the point (x,y,z) on the tangent plane that corresponds to $(x,y,f(x,y))$. The equation for the tangent plane at the surface is given by

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

The **linear approximation** of a function at point (x,y) is the value of the linearization at that point. So we have:

$$f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

NOTE: We can also find the equation of a tangent plane using a 3-variable linearization:

$$f(x,y,z) \approx L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

The equation of the plane is NOT the linearization, but instead would be:

$$f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0$$

NOTE: I go in much greater detail with example problems on the sample midterm, as well as on Handout 6 on my website.

For the second tangent plane equation PLEASE LOOK AT THE PRACTICE MIDTERM SOLUTIONS.

Chapter 14.5: Directional Derivatives and the Gradient

The gradient of f is a VECTOR FUNCTION given by:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

The gradient gives us the direction of **greatest increase from the point**. One of the implications of this is thatthe gradient will be **NORMAL to the level curves** of a function of two variables $f(x, y)$ and **NORMAL to the level surfaces** of a function of three variables $f(x, y, z)$. The second part is important because if you have some level surface $f(x, y, z) = c$, then the **EQUATION OF THE TANGENT PLANE AT POINT** (a, b, c) **is given by**:

$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0$$

where $\langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c)\rangle$ is the gradient at point (a, b, c). Recall: you can get the equation of a tangent plane with a NORMAL VECTOR and a POINT. The gradient is a normal vector to a level surface at a point, SO it can be your normal vector for your tangent plane.

The directional derivative at point $P = (a, b, c)$ and in the direction of some vector $\mathbf{v}$ is a NUMBER given by the formula:

$$D_{\mathbf{v}}(a, b, c) = \nabla f(a, b, c) \cdot \mathbf{e}_{\mathbf{v}}$$

Where $\mathbf{e}_{\mathbf{v}}$ is the unit vector of $\mathbf{v}$. The directional derivative gives us a number indicating the amount of CHANGE that occurs in the direction we are looking at. The closer the direction is to the gradient, the greater the value of the directional derivative. If the direction is closer to the opposite direction to the gradient, then it is going to be negative, and it is going to be a direction of DECREASE.

NOTE: I go in much greater detail with example problems on the sample midterm