Midterm I Solutions Math 20C, Summer Session I 2017

Name:

PID: Section ID/Time/TA: Seat:

DO NOT OPEN THE EXAM UNTIL THE INSTRUCTOR INDICATES EXAM START

The double bars $\|\|$ refer to the norm/magnitude/length of the vectors, as discussed in lecture.

Instructions:

1. The only notes you may use are a 3"x5" notecard or equally sized piece of paper with notes on them. The use of electronic devices, textbooks, or extra notes (beyond the allowed 3"x5" note card) is not allowed for the exam.
2. Write your name, PID, section, AND seat number on this cover page.
3. Read each question carefully, answer each question completely.
4. Indicate below whether you completed a problem on the provided scratch paper, and need us to consider the scratch paper to fully grade your exam. If you do not check "yes" we will assume that you do not need us to consider the scratch paper to fully grade your exam.
5. Show ALL of your work for each part. You are allowed to refer to your answers on previous parts of a question to solve a part in the same question. **CREDIT MAY NOT BE GIVEN FOR UNSUPPORTED ANSWERS**

Did you complete a problem on the provided scratch paper?

☐ YES ☐ NO

This exam has 5 questions. Questions 1, 2, 3, and 5 have two (2) parts. Question 4 has three (3) parts. The exam has content on both the front and back of sheets. In total, the exam is 10 pages, were each side of a sheet is a page. The first page is this cover page, and the last 3 pages (1.5 sheets) is scratch paper.

The midterm questions are divided into 3 types:

- Type 1 (Easy): These are problems that are just re-numberings of homework/quiz problems. The following parts or problems are Type 1: 1a, 2a, 4c, 5a
- Type 2 (Challenging): These are problems that combine homework concepts, but should otherwise look familiar. They would have the twist of using not just the idea behind one homework problem, but from multiple to reach a solution. The following parts or problems are Type 2: 1b, 4b, 3ab
- Type 3 (Hard): These are problems you should have never seen before that use concepts from the homework, quizzes, lecture, and readings. The following parts or problems are Type 3: 2b, 4a, 5b

Problem 1. (10 points)
a) Find the equation of the line of intersection between the planes $x+2y+3z=5$ and $x+2y-z=1$

This is a renumbering of homework problem 1.3 #20.
There are two ways to approach this problem.

Method 1:
Our goal is to find two points on the line and then use the point-point formula to find the equation on the line. One way to find two points is to fix a variable, like y to a specific value, and solve the resulting system of equations. If we set $y=0$, solutions to the resulting system of equations would be points (a,c), and the resulting points of intersection would be $(a,0,c)$, as y was set to 0. Say we set $y=0$. The resulting system is:

$$\begin{aligned}
&x+3z=5 \implies 4z=4 \implies z=1 \implies x-1=1 \implies x=2\\
&x-z=1
\end{aligned}$$

The first point of intersection is $(2,0,1)$. Next, we can try setting $y=1$. Then we have:

$$\begin{aligned}
&x+2+3z=5 \implies x+3z=3 \implies 4z=4 \implies z=1 \implies x-1=-1 \implies x=0\\
&x+2-z=1 \implies x-z=-1
\end{aligned}$$

The second point of intersection is $(0,1,1)$. Since the planes intersect in a line, and we have two points on that line, we can find the equation by taking the point-point form of the line equation. $(2,0,1)-(0,1,1)$ gives us a direction vector $(2,-1,0)$. We can then use either point for the point on the line, so we'll use $(0,1,1)$, and our final equations is:

$$\boxed{l(t)=(0,1,1)+t(2,-1,0)}$$

Method 2:

The second method involves the point-direction form of the line equation. You can find a single point shared by both planes, as in the first method. So taking $y=0$, we get $(2,0,1)$ as the point. Next we find a direction. We note that the line of intersection is contained on both planes, and must consequently have a direction that is perpendicular to the normal vectors of both planes. Hence, a cross product of the normal vectors would give a direction vector for the line.

$$\begin{vmatrix} i & j & k\\ 1 & 2 & 3\\ 1 & 2 & -1\end{vmatrix} = \begin{vmatrix} 2 & 3\\ 2 & -1\end{vmatrix} i - \begin{vmatrix} 1 & 3\\ 1 & -1\end{vmatrix} j + \begin{vmatrix} 1 & 2\\ 1 & 2\end{vmatrix} k = -8i+4j = (-8,4,0)$$

Our final equations is:

$$\boxed{l(t)=(2,0,1)+t(-8,4,0)}$$

b) Find the equation of a plane that contains the line $q(t)=(7,2,5)+t(2,4,0)$ and contains the point of intersection between the two lines $r(t)=(5,7,9)+t(4,5,6)$ and $s(t)=(3-2t,2,1+2t)$

The first part of this problem is to find a point of intersection between the lines: $r(t)=(5,7,9)+t(4,5,6)$ and $s(t)=(3-2t,2,1+2t)$. In doing so, we have a point on the plane, as well as a point needed to find two vectors that lie on the plane, and hence a normal vector. Let's say the parameter for $r(t)$ was t_1 and the parameter for $s(t)$ is t_2. So we want to find t_1 and t_2 such that $r(t_1) = s(t_2)$. We have separate parameters, because lines may not intersect at the same value of t.

$$r(t_1) = (5,7,9) + t_1(4,5,6) = (5+4t_1, 7+5t_1, 9+6t_1)$$
$$s(t_2) = (3-2t_2, 2, 1+2t_2)$$

We can set all components of the above two equations equal to each other to find the intersection point. so $5+4t_1 = 3-2t_2$, $7+5t_1 = 2$, and $9+6t_1 = 1+2t_2$. The easiest of these equations to solve is the middle one, since it has only one variable.

$$7+5t_1 = 2 \implies 5t_1 = -5 \implies t_1 = -1 \implies 5+4(-1) = 3-2t_2 \implies -2t_2 = -2 \implies t_2 = 1$$

We can then check to verify that these solutions result in the same point by comparing the results in the third equation:

$$9+6(-1) = 1+2(1) \implies 3 = 3$$

Or we can plug in $t_1 = -1$ and $t_2 = 1$ into the respective line equations.

$$r(-1) = (5+4(-1), 7+5(-1), 9+6(-1) = (1,2,3)$$
$$s(1) = (3-2(1), 2, 1+2(1)) = (1,2,3)$$

So the point on the plane is $(1,2,3)$

Next, we need to find a normal vector. This one is a bit trickier, and requires us to find three points on the plane (or two points, if you recognize that the direction vector of $q(t)$ lies on the plane). We have one point on the plane, and can select two others from the line $q(t)$ which also lies on the plane. So we can select $q(0) = (7,2,5)$ and $q(1) = (9,6,5)$ which are all points on the plane.

We can then take the differences of distinct pairs of points to find two vectors that lie on the plane. We'll take $\vec{u} = q(1) - q(0) = (9,6,5) - (7,2,5) = (2,4,0)$, and we'll take $\vec{v} = q(0) - (1,2,3) = (6,0,2)$. The two vectors $\vec{u} = (2,4,0)$ and $\vec{v} = (6,0,2)$ lie on the plane, so we may take the cross product to find the normal vector $\vec{n}$.

$$\vec{n} = \begin{vmatrix} i & j & k \\ 2 & 4 & 0 \\ 6 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} i - \begin{vmatrix} 2 & 0 \\ 6 & 2 \end{vmatrix} j + \begin{vmatrix} 2 & 4 \\ 6 & 0 \end{vmatrix} k = 8i - 4j - 24k = (8, -4, -24)$$

We now have both a point and a normal vector and may plug in those values into the plane equation formula. The result is:

$$\boxed{8(x-1) - 4(y-2) - 24(z-3) = 0}$$

Problem 2. (10 points) Let $\vec{u}$ and $\vec{v}$ be vectors where $\|\vec{u}\| = 5$, $\|\vec{v}\| = 6$, and $\|\vec{u} - \vec{v}\| = 9$.
a) Find $\|2\vec{u} + \vec{v}\|$.

This problem is just a renumbering of quiz 2, problem 1, which I explicitly stated was likely to appear on the exam. Kudos to those of you who paid attention to that. The main technique in this problem is to use the property $\|\vec{a}\|^2 = \vec{a} \cdot \vec{a}$. We begin by applying this to $\|\vec{u} - \vec{v}\|$

$$81 = \|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u}-2\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{v} = \|\vec{u}\|^2-2\vec{u}\cdot\vec{v}+\|\vec{v}\|^2 = 25-2\vec{u}\cdot\vec{v}+36 \implies \vec{u}\cdot\vec{v} = -10$$

From there, we reapply the same technique to the target magnitude $\|2\vec{u} + \vec{v}\|$.

$$\|2\vec{u}+\vec{v}\|^2 = (2\vec{u}+\vec{v})\cdot(2\vec{u}+\vec{v}) = 4\vec{u}\cdot\vec{u}+4\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{v} = 4\|\vec{u}\|^2+4\vec{u}\cdot\vec{v}+\|\vec{v}\|^2 = 4(5)^2+4(-10)+6^2 = 96$$

The final answer is the square root of the above result, $\sqrt{96} = \boxed{4\sqrt{6}}$

b) Find $\|\vec{u} \times \vec{v}\|$

This part involves a formula I explicitly told the class would be important during lecture 1 and 2, and that it would likely be on the exam (star). To do this problem, we note two things:

1. $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$
2. $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$

From here, we can find $\cos\theta$, and consequently, $\sin\theta$. From part a), we have:

$$\vec{u}\cdot\vec{v} = -10 = 5(6)\cos\theta \implies \cos\theta = -\frac{1}{3}$$

Since $\cos\theta = -\frac{1}{3}$, using the Pythagorean theorem, taking the adjacent leg to be -1 and the hypotenuse to be 3, we find that the opposite leg of the triangle is $\sqrt{8}$. Hence $\sin\theta = \frac{\sqrt{8}}{3}$. Note that since θ by convention is between 0 and π for angles between two vectors, $\sin\theta$ is non-negative for angles between two vectors. Also, please note that the ability to switch between trig functions is something I had specifically tested you all on on quiz 1, and expected you to know this coming in. Plugging in everything into the formula,

$$\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta = 5(6)\left(\frac{\sqrt{8}}{3}\right) = 10\sqrt{8} = \boxed{20\sqrt{2}}$$

Problem 3. (10 points) Let the acceleration of a particle be represented by $a(t) = (-4\cos t, 3\cos t, -5\sin t)$. Let the initial velocity be $v(0) = (3, 4, 5)$, and let the initial position on the path be $r(0) = (4, 0, 1)$

a) Find $r(t)$, the equation for the path described above.

The problem above was not considered a type 1 easy problem because technically none of the homework required you to integrate from acceleration to position. I did cover this in lecture, though. We integrate $a(t)$ to get $v(t)$, then integrate $v(t)$ to get $r(t)$.

$$\int a(t) = \left(\int -4\cos t, \int 3\cos t, \int -5\sin t\right) = (-4\sin t + C_1, 3\sin t + C_2, 5\cos t + C_3)$$

$$v(0) = (3,4,5) = (-4\sin(0)t + C_1, 3\sin(0) + C_2, 5\cos(0) + C_3) = (C_1, C_2, 5 + C_3) \implies (C_1, C_2, C_3) = (3, 4, 0)$$

So we have $v(t) = (-4\sin t + 3, 3\sin t + 4, 5\cos t)$, and we integrate $v(t)$ to get the position function $r(t)$.

$$\int v(t) = \left(\int -4\sin t + 3, \int 3\sin t + 4, \int 5\cos t\right) = (4\cos t + 3t + C_4, -3\cos t + 4t + C_5, 5\sin t + C_6)$$

$$r(0) = (4, 0, 1) = (4\cos(0) + 3(0) + C_4, -3\cos(0) + 4(0) + C_5, 5\sin(0) + C_6) = (4 + C_4, -3 + C_5, C_6)$$

This gives $(C_4, C_5, C_6) = (0, 3, 1)$. Hence our final answer is:

$$\boxed{r(t) = (4\cos t + 3t, -3\cos t + 4t + 3, 5\sin t + 1)}$$

b) Find the length of the path $r(t)$ from $t = \frac{\pi}{12}$ to $t = \frac{13\pi}{12}$.

We note that the formula for arc length, which I had gone over in lecture and which was in the homework, from $x = a$ to $x = b$ of a path $r(t) = (x(t), y(t), z(t))$ is:

$$\int_a^b \|v(t)\| = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

In our case, this gives us:

$$\begin{aligned}
\int_{\frac{\pi}{12}}^{\frac{13\pi}{12}} \|v(t)\| &= \int_{\frac{\pi}{12}}^{\frac{13\pi}{12}} \sqrt{(-4\sin t + 3)^2 + (3\sin t + 4)^2 + (5\cos t)^2} \\
&= \int_{\frac{\pi}{12}}^{\frac{13\pi}{12}} \sqrt{16\sin^2 t - 24\sin t + 9 + 9\sin^2 t + 24\sin t + 16 + 25\cos^2 t} \\
&= \int_{\frac{\pi}{12}}^{\frac{13\pi}{12}} \sqrt{25\sin^2 t + 25\cos^2 t + 25} = \int_{\frac{\pi}{12}}^{\frac{13\pi}{12}} 5\sqrt{\sin^2 t + \cos^2 t + 1} \\
&= \int_{\frac{\pi}{12}}^{\frac{13\pi}{12}} 5\sqrt{2} = 5\sqrt{2}t\Big|_{t=\frac{\pi}{12}}^{\frac{13\pi}{12}} = 5\sqrt{2}\left(\frac{13\pi}{12}\right) - 5\sqrt{2}\left(\frac{\pi}{12}\right) \\
&= 5\sqrt{2}\left(\frac{13\pi}{12} - \frac{\pi}{12}\right) = \boxed{5\sqrt{2}\pi}
\end{aligned}$$

Problem 4. (10 points) Let $f(x,y) = x^2y$, and let $g(x,y) = x^4 + y^2$.
a) Find

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{g(x,y)}$$

if it exists. Otherwise, if the limit does not exist, show that the limit does not exist.

The limit does not exist. To show this, we find the limit along two separate paths that intersect $(0,0)$. First, we can take $y = 0$.

$$\lim_{(x,0)\to(0,0)} \frac{x^2(0)}{x^4 + 0^2} = \lim_{(x,0)\to(0,0)} \frac{0}{x^4} = 0$$

Then we may take the limit along $y = x^2$.

$$\lim_{(x,x^2)\to(0,0)} \frac{x^2(x^2)}{x^4 + (x^2)^2} = \lim_{(x,x^2)\to(0,0)} \frac{x^4}{x^4 + x^4} = \lim_{(x,x^2)\to(0,0)} \frac{x^4}{2x^4} = \frac{1}{2}$$

Since the limits from two different paths are different, the limit itself must not exist.

b) Let P be the tangent plane to $g(x,y)$ at the point $(1,2,5)$. Find all points on the graph of f that have tangent plane parallel to P.

This problem is similar to a homework problem. We must first find the equation of the tangent plane to $g(x,y)$ at the point $(1,2,5)$. This plane equation is given by:

$$z = g(1,2) + \frac{\partial g}{\partial x}(1,2)(x-1) + \frac{\partial g}{\partial y}(1,2)(y-2)$$

$g(1,2) = 1^4 + 2^2 = 5$. We then find $\frac{\partial g}{\partial x}(x,y) = 4x^3$, plug in $(1,2)$, and get $\frac{\partial g}{\partial x}(1,2) = 4(1)^3 = 4$. Then we find $\frac{\partial g}{\partial y} = 2y$, plug in $(1,2)$, and get $\frac{\partial g}{\partial y}(1,2) = 2(2) = 4$. The initial plane equation we have is:

$$z = 5 + 4(x-1) + 4(y-2) \implies 4(x-1) + 4(y-2) - 1(z-5) = 0$$

We convert the equation above to a more standard form to extract the normal vector to $g(x,y)$ at the point indicated, and that normal vector is $(4,4,-1)$. In order for two planes to be parallel, their normal vectors must also be parallel. To indicate the points on the graph of f where our original tangent plane is parallel, we would want to know what the tangent planes of f look like at any point (a,b). They would look like:

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) \implies \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) - 1(z - f(a,b)) = 0$$

We convert the equation above to a more standard form to extract the normal vector to $f(x,y)$ at any point (a,b), $(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b), -1)$
In order for the normal vectors to be parallel, we need:

$$(4,4,-1) = k(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b), -1) = (k\frac{\partial f}{\partial x}(a,b), k\frac{\partial f}{\partial y}(a,b), -k) \implies -k = -1 \implies k = 1$$

Therefore, we only need:

$$(4,4,-1) = (\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b), -1) \implies \frac{\partial f}{\partial x}(a,b) = 4, \frac{\partial f}{\partial y}(a,b) = 4$$

We find the partial derivatives of f, which are $\frac{\partial f}{\partial x}(x,y) = 2xy$ and $\frac{\partial f}{\partial y}(x,y) = x^2$. We can then find the partial (a,b).

$$\frac{\partial f}{\partial y}(a,b) = a^2 = 4 \implies a = 2 \text{ or } a = -2$$

$$\frac{\partial f}{\partial y}(2,b) = 2(2)y = 4y = 4 \implies y = 1 \text{ and } \frac{\partial f}{\partial y}(-2,b) = 2(-2)y = -4y = 4 \implies y = -1$$

The two points we have are (a,b) are $(2,1)$ and $(-2,-1)$. To find the points ON THE GRAPH of f, we need points in the form $(a,b,f(a,b))$. So $f(2,1) = 2^2(1) = 4$ and $f(-2,-1) = (-2)^2(-1) = -4$. The points on the graph are $\boxed{(2,1,4) \text{ and } (-2,-1,-4)}$.

c) Using linear approximation to approximate the function g(x,y), estimate $(1.0012)^4 + (2.005)^2$

Linear approximation in the case of scalar-valued functions of two variables, amounts to using the equation of the tangent plane to approximate points close to an easily calculated point. In this case, the easy to calculate point is $(1,2)$, and we use the tangent plane of $g(x,y)$ at $(1,2)$, which should have been computed in part a).

$$z = 5 + 4(x-1) + 4(y-2)$$

We plug in the point we're trying to approximate, $(1.0012, 2.005)$. The resulting z value gives the approximation of the value of g at that point.

$$z = 5 + 4(1.0012 - 1) + 4(2.005 - 2) = 5 + 4(0.0012) + 4(0.005) = 5 + 0.0048 + 0.02 = \boxed{5.0248}$$

Problem 5. (10 points) **A princess named Jen is walking around a flat field when Moltres, a legendary fire bird appears to drastically change the temperature of the field. The new temperature, in Celsius, of the field is given by**

$$T(x, y) = \frac{100}{(x-2)^2 + 4(y-1)^2 + 2} + 30$$

where x and y are measured in meters.

a) Jen is standing at point $(4, 2)$ and is attempting to flee from the heat. If Jen is moving at unit speed (1 meter per second) from the point $(4, 2)$, in what direction should she run if she wants to run in the direction of fastest temperature decrease? Express the direction of decrease as a vector. What is the directional derivative along the direction of fastest temperature decrease?

This problem is like 2.6 # 22ab, but with unit speed. What is important to know for this problem is that the direction of greatest **decrease** is the direction opposite of the gradient. So at any point (a, b), the direction of greatest decrease of T is $-\nabla T(a, b)$.

$$\begin{aligned}
-\nabla T(x, y) &= -\left(-\frac{100}{\left((x-2)^2 + 4(y-1)^2\right)^2} \cdot 2(x-2), -\frac{100}{\left((x-2)^2 + 4(y-1)^2\right)^2} \cdot 8(y-1)\right) \\
&= \frac{100}{\left((x-2)^2 + 4(y-1)^2\right)^2} \left(2(x-2), 8(y-1)\right) \\
-\nabla T(4, 2) &= \frac{100}{\left((4-2)^2 + 4(2-1)^2\right)^2} \left(2(4-2), 8(2-1)\right) = \frac{100}{10^2} \left(2(2), 8(1)\right) = \boxed{(4, 8)}
\end{aligned}$$

The rate of change in this direction, since Jen is moving at unit speed, is equivalent to the directional derivative in the direction opposite of the gradient. There are two ways to compute this. The first is the standard formula, where you take the dot product of the gradient with a UNIT VECTOR in the direction of greatest decrease. We have that the gradient at the point is $\nabla T(4, 2) = (-4, -8)$ from above. Now we need a unit vector in the direction of (4,8). The unit vector of $(4, 8)$ can be computed by dividing $(4, 8)$ by its magnitude. $\|(4, 8)\| = \sqrt{4^2 + 8^2} = \sqrt{80} = 4\sqrt{5}$. The resulting unit vector is $\frac{(4,8)}{4\sqrt{5}} = (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$.
The resulting directional derivative is:

$$D_{(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})} T(4, 2) = (-4, -8) \cdot (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}) = -4 \cdot \frac{1}{\sqrt{5}} - 8 \cdot \frac{2}{\sqrt{5}} = -\frac{20}{\sqrt{5}} = \boxed{-4\sqrt{5}}$$

The second is to note that the opposite direction forms an angle of π radians with the gradient and use the alternate formula for the directional derivative.

$$D_{(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})} T(4, 2) = \|\nabla T(4, 2)\| cos\pi = \boxed{-4\sqrt{5}}$$

b) After fleeing, Jen returns with a heat-resistant suit and enough water to fight the fire bird. She is dropped at the point (0,0) in search of the fire bird. Unfortunately, the heat-resistant suit does not like change and will disintegrate if it is heated at a rate greater than $2\sqrt{15}$ degrees per second, or if it is cooled down at a rate greater than $2\sqrt{5}$ degrees per second.

If Jen is moving at unit speed (1 meter per second), from the point (0,0), what is the minimum angle and maximum angle from the gradient that Jen can move in order to prevent the suit from disintegrating?

Naturally, just as before, the gradient is important. The gradient is the direction of fastest increase, and is something used to compute the rate of change at a point. In the case of unit speed, as in this problem, the rate of change is equal to the directional derivative. So we first compute the gradient as in part a), but at point (0,0).

$$\begin{aligned}
\nabla T(x,y) &= \left(-\frac{100}{\left((x-2)^2+4(y-1)^2\right)^2}\cdot 2(x-2), -\frac{100}{\left((x-2)^2+4(y-1)^2\right)^2}\cdot 8(y-1)\right) \\
&= -\frac{100}{\left((x-2)^2+4(y-1)^2\right)^2}\left(2(x-2), 8(y-1)\right) \\
\nabla T(0,0) &= -\frac{100}{\left((0-2)^2+4(0-1)^2\right)^2}\left(2(0-2), 8(0-1)\right) = -\frac{100}{10^2}\left(2(-2), 8(-1)\right) = (4,8)
\end{aligned}$$

It is beneficial to use the alternate formula for the directional derivative here, since that formula actually involves angles between the gradient and the direction of choice.

$$D_{\vec{v}}T(0,0) = \|\nabla T(0,0)\| cos\theta = \|(4,8)\|\cos\theta = 4\sqrt{5}\cos\theta$$

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We want the rate of change to be no less than $-2\sqrt{5}$, since we don't want cooling at a greater rate. We also want the rate of change to be no greater than $2\sqrt{15}$, because we don't want the heating at a greater rate. Therefore:

$$-2\sqrt{5} \le 4\sqrt{5}\cos\theta \le 2\sqrt{15} \implies -\frac{1}{2} \le \cos\theta \le \frac{\sqrt{3}}{2}$$

When looking at the unit circle between 0 and π, we find that $\cos\theta \le \frac{\sqrt{3}}{2}$ when $\theta \ge \frac{\pi}{6}$ and also $\cos\theta \ge -\frac{1}{2}$ when $\theta \le \frac{2\pi}{3}$

So the minimum angle is $\boxed{\frac{\pi}{6}}$ and the maximum angle is $\boxed{\frac{2\pi}{3}}$

SCRATCH PAPER

SCRATCH PAPER

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