Different forms of Linear Systems, Linear Combinations, and Span (1.3-1.4)

Terms you should know

**Linear combination (and weights):** A vector $y$ is called a *linear combination* of vectors $v_1, v_2, ..., v_k$ if given some numbers $c_1, c_2, ..., c_k$,

$$ y = c_1 v_1 + c_2 v_2 + ... + c_k v_k $$

The numbers $c_1, c_2, ..., c_k$ are called weights.

**Span:** We call the set of ALL the possible linear combinations of a set of vectors to be the span of those vectors. For example, the span of $v_1$ and $v_2$ is written as $\text{span}(v_1, v_2)$

**NOTE:** The zero vector is in the span of ANY set of vectors in $\mathbb{R}^n$.

$\mathbb{R}^n$: The set of all vectors with $n$ entries

3 ways to write a linear system with $m$ equations and $n$ unknowns

If $A$ is the $m \times n$ coefficient matrix corresponding to a linear system, with columns $a_1, a_2, ..., a_n$, and $b$ in $\mathbb{R}^m$ is the constant vector corresponding to that linear system, we may represent the linear system using

1. A matrix equation: $Ax = b$
2. A vector equation: $x_1 a_1 + x_2 a_2 + ... + x_n a_n = b$, where $x_1, x_2, ... x_n$ are numbers.
3. An augmented matrix: $[ A | b ]$

**NOTE:** You should know how to multiply a matrix by a column vector, and that doing so would result in some linear combination of the columns of that matrix.
Important theorems to know:

**Theorem.** (Chapter 1, Theorem 3) If $A$ is an $m \times n$ matrix, with columns $a_1, a_2, ..., a_n$ and $b$ is in $\mathbb{R}^m$, the matrix equation

$$Ax = b$$

has the same solution set as the vector equation

$$x_1a_1 + x_2a_2 + ... + x_na_n = b$$

as well as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} A & b \end{bmatrix}$$

**Theorem.** $b$ can be written as a linear combination of the columns of $A$ (aka $b$ is in the span of the columns of $A$) if and only if $Ax = b$ has a solution.

**Theorem.** (Chapter 1, Theorem 4) If $A$ is an $m \times n$ matrix, with columns $a_1, a_2, ..., a_n$. The following are equivalent (so they’re either all true or all false)

1. For each and every $b$ in $\mathbb{R}^m$, $Ax = b$ has a solution
2. For each and every $b$ in $\mathbb{R}^m$, $x_1a_1 + x_2a_2 + ... + x_na_n = b$ has a solution, i.e. $b$ is a linear combination of the columns of $A$.
3. The columns of $A$ span $\mathbb{R}^m$. Alternatively, we say that $\mathbb{R}^m$ is in the span of the columns of $A$, and $\text{span(columns of }A) = \mathbb{R}^m$
4. $A$ has a pivot position in every row.

**NOTE:** When talking about linear systems, $A$ refers to the coefficient matrix, NOT the augmented matrix

**Theorem.** (Chapter 1, Theorem 5) If $A$ is an $m \times n$ matrix, and $u$ and $v$ are vectors in $\mathbb{R}^n$, and $c$ is some number.

1. $A(u + v) = Au + Av$
2. $A(cu) = c(Au)$

This is known as the **linearity** property.