1 Linear systems, existence, uniqueness

For each part, construct an augmented matrix for a linear system with the given properties, then give the corresponding vector equation and matrix equation for the system:

a) A 4x3 system with no solution

b) A 4x4 system with infinitely many solutions

c) A 5x4 system with one unique solution

Solution:

The key theorem for this problem is chapter 1, theorem 2. A system is inconistent (has no solution) IF AND ONLY IF the reduced row echelon form of the AUGMENTED MATRIX for the system has a row with a leading entry in the last column, i.e. you have a row in the augmented matrix that looks like:

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & x \end{bmatrix}, \quad x \neq 0$$

OTHERWISE, the system is consistent (has at least one solution). Also, if the system is CONSISTENT and has NO FREE VARIABLES, it has one unique solution. If it is CONSISTENT and has AT LEAST ONE free variable, it has infinitely many solutions.

KEEP IN MIND: there are infinitely many ways to go about solving this problem. I'll give you an RREF and a non-echelon form solution (that is row equivalent to the RREF) for each part to give you an idea.

Solution to part a)

We have a 4×3 system with NO solution. So what we know is that there are 4 equations and 3 unknowns. That means the AUGMENTED MATRIX has 4 rows and 4 columns. We also know that the system has no solution, so it is inconsistent. That means the reduced row echelon form of the matrix has a row that looks like the above.

$$\text{RREF: } \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad \text{NOT RREF: } \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right)$$

Then we give the vector equations:

$$\text{from RREF matrix: } x_1 \begin{bmatrix}1\\0\\0\\0\end{bmatrix} + x_2 \begin{bmatrix}0\\1\\0\\0\end{bmatrix} + x_3 \begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$\text{from NOT RREF matrix: } x_1 \begin{bmatrix}1\\0\\0\\1\end{bmatrix} + x_2 \begin{bmatrix}0\\1\\0\\1\end{bmatrix} + x_3 \begin{bmatrix}0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

And the matrix equation:

$$\text{from RREF matrix: } \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix} x = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \qquad \text{from NOT RREF matrix: } \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\end{bmatrix} x = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Solution to part b)

We have a 4×4 system with infinitely many solutions. So what we know is that there are 4 equations and 4 unknowns. That means the AUGMENTED MATRIX has 4 rows and 5 columns. We also know that there are infintely many solutions. **So that means 1) it is consistent and 2) it has at least one free variable**

$$\text{RREF: } \left(\begin{array}{cccc|c}1&-1&0&0&5\\0&0&1&0&6\\0&0&0&1&7\\0&0&0&0&0\end{array}\right) \qquad \text{NOT RREF: } \left(\begin{array}{cccc|c}1&-1&0&0&5\\0&0&1&0&6\\0&0&0&1&7\\1&-1&1&1&18\end{array}\right)$$

Then we give the vector equations:

$$\text{from RREF matrix: } x_1 \begin{bmatrix}1\\0\\0\\0\end{bmatrix} + x_2 \begin{bmatrix}-1\\0\\0\\0\end{bmatrix} + x_3 \begin{bmatrix}0\\1\\0\\0\end{bmatrix} + x_4 \begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}5\\6\\7\\0\end{bmatrix}$$

$$\text{from NOT RREF matrix: } x_1 \begin{bmatrix}1\\0\\0\\1\end{bmatrix} + x_2 \begin{bmatrix}-1\\0\\0\\-1\end{bmatrix} + x_3 \begin{bmatrix}0\\1\\0\\1\end{bmatrix} + x_4 \begin{bmatrix}0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}5\\6\\7\\18\end{bmatrix}$$

And the matrix equation:

$$\text{from RREF matrix: } \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 0 \end{bmatrix}$$

$$\text{from NOT RREF matrix: } \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 18 \end{bmatrix}$$

Solution to part c)

We have a 5×4 system with one unique solution. So what we know is that there are 5 equations and 4 unknowns. That means the AUGMENTED MATRIX has 5 rows and 5 columns. We also know that there is ONE unique solution. **So that means 1) it is consistent and 2) it has no free variables**

$$\text{RREF: } \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \text{NOT RREF: } \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 1 & 1 & 1 & 1 & 10 \end{array}\right)$$

Then we give the vector equations:

$$\text{from RREF matrix: } x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 0 \end{bmatrix}$$

$$\text{from NOT RREF matrix: } x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 10 \end{bmatrix}$$

And the matrix equation:

$$\text{from RREF matrix: } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 0 \end{bmatrix}$$

$$\text{from NOT RREF matrix: } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 10 \end{bmatrix}$$

ALSO, a key idea to keep in mind for this exam and for the rest of the course is the ability to switch between augmented matrix, to vector equation, to matrix equation. Those problems are equivalent and have the same solution sets.

2 Linear Combinations and Span

Let $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Does the set $\{u, v, w\}$ span $\mathbb{R}^2$? If so, justify your answer. Otherwise, find a vector that is not in span(u, v, w)

Solution:

The key here is to know what span means and what it means to span $\mathbb{R}^2$. Now a set of vectors spans $\mathbb{R}^2$ if every vector in $\mathbb{R}^2$is a linear combination of the vectors in the set. In other words, for any numbers h and k, we have

$$x_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}$$

But this is the same as saying that the augmented matrix below corresponds to a consistent system:

$$\left(\begin{array}{ccc|c} 2 & 2 & 1 & h \\ -1 & 1 & 1 & k \end{array}\right)$$

So we row reduce by adding half of the first row to the second row:

$$\left(\begin{array}{ccc|c} 2 & 2 & 1 & h \\ -1 & 1 & 1 & k \end{array}\right) \implies \left(\begin{array}{ccc|c} 2 & 2 & 1 & h \\ 0 & 2 & \frac{3}{2} & k + \frac{h}{2} \end{array}\right)$$

Now notice that there is a pivot in every row of the coefficient matrix (the matrix that is formed from the elements left of the bar):

$$\left(\begin{array}{ccc|c} \boxed{2} & 2 & 1 & h \\ 0 & \boxed{2} & \frac{3}{2} & k + \frac{h}{2} \end{array}\right)$$

That means the system corresponding to the echelon form has a solution, which means the original system has a solution, which means that no matter what you choose for h and k, the system has a solution.

So every vector in $\mathbb{R}^2$ is a linear combination of the vectors u, v, and w, and the set spans $\mathbb{R}^2$

3 Homogenous Systems, Independence and Dependence

a) Let B be the following matrix:

$$B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 5 & 9 \\ 0 & 7 & 10 \end{bmatrix}$$

Solve the homogeneous system $Bx = 0$. Are the columns of B linearly independent? Give a description of the span of the columns of B.

b) Suppose $\{x, y\}$ is a linearly independent set and $\{x, y, z\}$ is a linearly dependent set. Is $z \in \text{span}(x, y)$? If so, prove that $z \in \text{span}(x, y)$. Otherwise, provide a counterexample.

c) Let A be the following matrix:

$$A = \begin{bmatrix} 42 & 221 & 1999 & 2015 \\ 9001 & \pi & \sqrt{3} & \frac{1}{3} \\ 24601 & 87 & \frac{\pi}{e} & \sin(\ln(16)) \end{bmatrix}$$

Are the columns of A linearly independent? Justify your answer.

Solution:

Solution to part a)

First off, you are asked to solve the homogeneous system $Bx = 0$. So you can do this by converting to an augmented matrix and row-reducing:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & 5 & 9 & 0 \\ 0 & 7 & 10 & 0 \end{array}\right)$$

$$(\text{row } 2 + \text{row } 1) \implies \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 7 & 12 & 0 \\ 0 & 7 & 10 & 0 \end{array}\right)$$

$$(\text{row } 3 + \text{row } 2) \implies \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 7 & 12 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right)$$

Now we notice that there is a pivot position in every row and a pivot position in every column of the coefficient matrix of this echelon form (and therefore in the original matrix):

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{7} & 12 & 0 \\ 0 & 0 & \boxed{-2} & 0 \end{array}\right)$$

Since there is a pivot in every column of the coefficient matrix, there are no free variables, meaning that if the system is consistent, there is only one solution. But it's the homogeneous system, which is always consistent, and always has the zero solution.

But that means the solution to $Bx = 0$ is the zero vector/ trivial solution.

Now there are two follow-up questions. One asks whether or not the columns of B are linearly independent.

Since the homogeneous system has ONLY the trivial solution, the columns of B are linearly independent.

Finally, the question asks to describe the span of the columns of B. But since the vectors are in $\mathbb{R}^3$, and there is a pivot in every row of the COEFFICIENT matrix, the system $Bx = b$ is always consistent for any choice of b. **So the span is all of $\mathbb{R}^3$.**

Solution to part b)

So what's important here is to know the definitions of span, linear independence, and linear dependence. Also, you should know how to approach a proof. So first off, it is $\boxed{true}$ that z is in $span(x, y)$. Buy why?

You want to show that z is a linear combination of x and y, and there are several ways to do this. But every way will exploit the other two facts: $\{x, y\}$ is a linearly **IN**dependent set and $\{x, y, z\}$ is a linearly dependent set.

So since $\{x, y, z\}$ is DEPENDENT, at least one of those vectors is a linear combination of the others. So AT LEAST one of the following are true for some numbers a and b:

1. $z = ax + by$
2. $x = ay + bz$
3. $y = ax + bz$

Now if $b \neq 0$, we can solve for z in the second and third equations.

1. $z = ax + by$
2. $x = ay + bz \implies z = \frac{1}{b}x - \frac{a}{b}y$
3. $y = ax + bz \implies z = \frac{1}{b}y - \frac{a}{b}x$

In which case, z would be a linear combination of x and y in every case. But we can only do this if $b \neq 0$. So we need to show that $b \neq 0$. But if $b = 0$, we have for the second and third equations:

1. $x = ay + bz \implies x = ay$
2. $y = ax + bz \implies y = ax$

But that means either x is a multiple of y or y is a multiple of x. But they're linearly **IN**dependent, so that's impossible. So we have necessarily, $b \neq 0$. Which means at least one of the following are true:

1. $z = ax + by$
2. $z = \frac{1}{b}x - \frac{a}{b}y$
3. $z = \frac{1}{b}y - \frac{a}{b}x$

But they ALL say that z is a linear combination of x and y, so z is in the span of x and y

Solution to part c)

The first instinct may be to row reduce this matrix, but you will quickly find that to be a terrible idea. To make life easier for you, you can just apply one of the theorems from the lecture, which in the textbook is chapter 1, theorem 8. The columns of the matrix are in $\mathbb{R}^3$, but there are 4 columns. $4 > 3$, so

$\boxed{\textbf{the columns are linearly dependent}}$

4 Linearity of Matrices

Suppose that v is a soution to the system $Ax = b_1$ and w is a solution to $Ax = b_2$. Show that the matrix equation $Ax = 42b_1 - \pi b_2$ has a solution.

Solution:

So the key play is to exploit the linearity property of matrices (Chapter 1, theorem 5). There are a couple ways to this one, but let's start with $42b_1 - \pi b_2$.
$42b_1 - \pi b_2 = 42Av - \pi Aw$ by substitution.
Next, we use the linearity property twice:

$$42b_1 - \pi b_2 = 42Av - \pi Aw = A(42v) + A(-\pi w) = A(42v - \pi w)$$

What this means is that $42v - \pi w$ is a solution to the system $Ax = 42b_1 - \pi b_2$. But that means the system has a solution, which is what you're trying to prove.

5 Matrix Inverses

$$\text{Let } A = \begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix}$$

Find a matrix B such that $AB = I$, where I is the Identity Matrix (show your work). Find BA. Is A invertible?

Solution:

We want to find B so that AB is I, the identity matrix. But $AB = A[\ b_1\ b_2\ ] = [\ Ab_1\ Ab_2\ ]$
To have the product atch the identity, we need to find b_1 so that $Ab_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and b_2 so that $Ab_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. But this is equivalent to solving two linear systems. So all we need to do is row reduce the augmented matrices, and solve each system:
For the first system:

$$\left(\begin{array}{cc|c} 7 & 5 & 1 \\ 4 & 3 & 0 \end{array}\right) \text{row 2 - } \tfrac{4}{7} \text{ row 1} \implies \left(\begin{array}{cc|c} 7 & 5 & 1 \\ 0 & \frac{1}{7} & -\frac{4}{7} \end{array}\right) \implies x_2 = -4, x_1 = 3$$

And for the second system:

$$\left(\begin{array}{cc|c} 7 & 5 & 0 \\ 4 & 3 & 1 \end{array}\right) \text{row 2 - } \tfrac{4}{7} \text{ row 1} \implies \left(\begin{array}{cc|c} 7 & 5 & 0 \\ 0 & \frac{1}{7} & 1 \end{array}\right) \implies x_2 = 7, x_1 = -5$$

And then we put it together to find:

$$B = \begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix}$$

When you multiply the matrices in reverse order, i.e. BA, you find that $BA = I$. Hence, $\boxed{\textbf{the matrix } A \textbf{ is invertible}}$.