2. Let $x, y \in X$ and suppose that $x \neq y$. Then $\{x\}^c$ is open in the cofinite topology and contains $y$ but not $x$. The cofinite topology on $X$ is therefore $T_1$.

Since $X$ is infinite it contains two distinct points $x$ and $y$. Suppose there exist disjoint open sets $A$ and $B$ (in the cofinite topology) such that $x \in A$ and $y \in B$. Then $A \subseteq B^c$, which is finite so $A$ is also finite. This is a contradiction because $A^c$ is finite but $X = A \cup A^c$ is infinite. Hence, the cofinite topology on $X$ is not $T_2$.

Suppose that $X$ is countable, and let $x \in X$. Choose a surjection $q : \mathbb{N} \to \{x\}$, and for each $n \in \mathbb{N}$ set $A_n := \{q(k)\}_{k=1}^n$ and $B_n := A_n^c$, so that $x \in B_n$ and $B_n$ is open. If $U$ is open and $x \in U$ then $U^c \subseteq \{x\}$ is finite and hence $U^c \subseteq A_n$ for some sufficiently large $N \in \mathbb{N}$. This implies that $B_N \subseteq U$, so $\{B_n\}_{n=1}^\infty$ is a neighbourhood base at $x$. The cofinite topology on $X$ is therefore first countable.

Conversely, suppose that the cofinite topology on $X$ is first countable. Choose $x \in X$ and a countable neighbourhood base $\{A_n\}_{n=1}^\infty$ at $x$. If $y \in \{x\}$ then $\{y\}^c$ is an open neighbourhood of $x$, so there exists $n \in \mathbb{N}$ such that $A_n \subseteq \{y\}$, and hence $y \in A_n^c$. It follows that $\{x\} \subseteq \bigcup_{n=1}^\infty A_n^c$, which is countable because each $A_n^c$ is finite. Therefore $X = \{x\} \cup \{x\}^c$ is also countable.

3. Let $(X, \rho)$ be a metric space with closed subspaces $A$ and $B$. Given $x \in X$ and $\varepsilon \in (0, \infty)$, let $y \in B_\varepsilon(x)$. If $a \in A$ then $\rho(x, A) \leq \rho(x, a) \leq \rho(x, y) + \rho(y, a) < \rho(y, a) + \varepsilon$, so $\rho(x, A) - \varepsilon \leq \rho(y, a)$. Similarly $\rho(y, A) - \varepsilon \leq \rho(x, A)$, so $- \varepsilon \leq \rho(y, A) - \rho(x, A) \leq \varepsilon$. This shows that $x \mapsto \rho(x, A)$ is continuous. Similarly $x \mapsto \rho(x, B)$ is continuous.

It follows that $x \mapsto \rho(x, A) - \rho(x, B)$ is continuous, so the preimages of $(-\infty, 0)$ and $(0, \infty)$ under this map are open. These preimages are $\{x \in X \mid \rho(x, A) < \rho(x, B)\}$ and $\{x \in X \mid \rho(x, A) > \rho(x, B)\}$, which are clearly disjoint. The first set contains $A$ and the second contains $B$ because $\rho(x, A) = 0$ iff $x \in A$ and $\rho(x, B) = 0$ iff $x \in B$. Indeed, if $x \in X$ and $\rho(x, A) = 0$ then every neighbourhood of $x$ meets $A$, because no $r \in (0, \infty)$ is a lower bound for $\{\rho(x, a)\}_{a \in A}$. Finally, $(X, \rho)$ is $T_1$ because distinct points are separated by positive distance. Therefore $(X, \rho)$ is normal.

4. Firstly $\emptyset, \mathbb{R} \in \mathcal{T}$ because $\emptyset = \emptyset \cup (\emptyset \cap \mathbb{Q})$ and $\mathbb{R} = \mathbb{R} \cup (\emptyset \cap \mathbb{Q})$. If $\{W_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ and $W_\alpha$ has decomposition $U_\alpha \cup (V_\alpha \cap \mathbb{Q})$ for each $\alpha \in A$, then $\cup_{\alpha \in A} W_\alpha = (\cup_{\alpha \in A} U_\alpha) \cup ((\cup_{\alpha \in A} V_\alpha) \cap \mathbb{Q}) \subseteq \mathcal{T}$. Moreover, if $A = \{1, 2\}$ then

$$W_1 \cap W_2 = (U_1 \cap (V_1 \cap \mathbb{Q})) \cap (U_2 \cup (V_2 \cap \mathbb{Q}))$$

$$= (U_1 \cup U_2) \cup (U_1 \cap (V_2 \cap \mathbb{Q})) \cup ((V_1 \cap \mathbb{Q}) \cap U_2) \cup ((V_1 \cap \mathbb{Q}) \cap (V_2 \cap \mathbb{Q}))$$

$$= (U_1 \cup U_2) \cup ((U_1 \cap V_2) \cup (V_1 \cap U_2) \cup (V_1 \cup V_2)) \cap \mathbb{Q}$$

$$\in \mathcal{T}.$$ 

By induction it follows that $\mathcal{T}$ is closed under finite intersections, so $\mathcal{T}$ is a topology.

Let $x, y \in \mathbb{R}$ be distinct. There exist disjoint open intervals $U, V \subseteq \mathbb{R}$ such that $x \in U$ and $y \in V$. Since $U = U \cup (\emptyset \cap \mathbb{Q})$ and $V = V \cup (\emptyset \cap \mathbb{Q})$, both these intervals are open in $\mathcal{T}$. Therefore $\mathcal{T}$ is Hausdorff.

Note that $\mathbb{Q}^c$ is closed in $\mathcal{T}$. Let $W_1, W_2 \in \mathcal{T}$ be neighbourhoods of 0 and $\mathbb{Q}^c$ respectively. Since 0 $\in W_1$, there exists $n \in \mathbb{N}$ such that $(-\frac{1}{n}, \frac{1}{n}) \cap \mathbb{Q} \subseteq W_1$. If $W_2$ has decomposition $U_2 \cup (V_2 \cap \mathbb{Q})$, then $\mathbb{Q}^c \subseteq U_2$ because $(V_2 \cap \mathbb{Q}) \cap \mathbb{Q}^c = \emptyset$. In particular $U_2 \cap (-\frac{1}{n}, \frac{1}{n}) \neq \emptyset$, because it contains, say, $\sqrt{2}/2n$. This implies that $U_2 \cap (-\frac{1}{n}, \frac{1}{n}) \emptyset$, since Q is dense in $\mathbb{R}$ with respect to the usual topology. Therefore $W_2 \cap W_1 \neq \emptyset$, so $\mathcal{T}$ is not regular.

5. Let $X$ be a separable metric space with a countable dense subspace $A$. Then $\mathcal{B} := \{B_{1/n}(a) \mid a \in A, n \in \mathbb{N}\}$ is countable because $\mathbb{N} \times \mathbb{N}$ is countable. Clearly each member of $\mathcal{B}$ is open in $X$. Conversely, if $U \subseteq X$ is open and $x \in U$, then $x \in B_{1/n}(x) \subseteq U$ for some $n \in \mathbb{N}$. If $x \in A$ then $B_{1/n}(x) \in \mathcal{B}$. Otherwise $x \in \text{acc}(A)$ because
X = \overline{A} = A \cup \text{acc}(A). It follows that \( B_{1/2n}(x) \) contains some point \( a \in A \), in which case \( x \in B_{1/2n}(a) \in \mathcal{B} \). By the triangle inequality \( B_{1/2n}(a) \subseteq B_{1/n}(x) \subseteq U \). This shows that \( U \) is the union of a (possibly empty) subcollection of \( \mathcal{B} \). Therefore \( \mathcal{B} \) is a base for the topology on \( X \), so this topology is second countable.

6. (a) Clearly \( \mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n) \). Let \( a, b, c, d \in \mathbb{R} \) and suppose that \( a < b \) and \( c < d \). Then \( (a, b] \cap (c, d] = \emptyset \) or \( \max\{a, c\} < \min\{b, d\} \), in which case \( (a, b] \cap (c, d] = (\max\{a, c\}, \min\{b, d\}) \). This shows that \( \mathcal{E} \) is a base for a topology \( \mathcal{T} \) on \( \mathbb{R} \). Clearly each member of \( \mathcal{E} \) is open in \( \mathcal{T} \). If \( a, b \in \mathbb{R} \) and \( a < b \) then

\[
(\bigcup_{n=1}^{\infty}(a - n, a - n + 1)) \cup (\bigcup_{n=1}^{\infty}(b + n - 1, b + n))
\]

is open in \( \mathcal{T} \), so its complement \((a, b]\) is closed in \( \mathcal{T} \).

(b) Let \( x \in \mathbb{R} \) and note that \( \mathcal{N} := \{(x - \frac{1}{n}, x)\}_{n=1}^{\infty} \) is a collection of open neighbourhoods of \( x \). If \( U \in \mathcal{T} \) contains \( x \) then (since \( \mathcal{E} \) is a base for \( \mathcal{T} \)) there exist \( a, b \in \mathbb{R} \) such that \( a < b \) and \( x \in (a, b] \subseteq U \). It follows that \( (x - \frac{1}{n}, x) \subseteq U \) for some \( n \in \mathbb{N} \) (with \( \frac{1}{n} \leq x - a \)). Therefore \( \mathcal{N} \) is a countable neighbourhood base at \( x \), so \( \mathcal{T} \) is first countable.

Let \( \mathcal{B} \) be a base for \( \mathcal{T} \), and let \( x \in \mathbb{R} \). Then \( \mathcal{B} \) contains a neighbourhood base at \( x \), so there exists \( U_x \in \mathcal{B} \) such that \( x \in U_x \subseteq (x - 1, x] \). In particular \( \sup U_x = x \), which implies that \( x \mapsto U_x \) is injective. Therefore \( \mathcal{B} \) is uncountable. This implies that \( \mathcal{T} \) is not second countable.

(c) Let \( x \in \mathbb{R} \) and let \( U \in \mathcal{T} \) contain \( x \). Then \( x \in (a, b] \subseteq U \) for some \( a, b \in \mathbb{R} \) with \( a < b \). There exists \( q \in (a, x) \cap \mathbb{Q} \), since \( \mathbb{Q} \) is dense in \( \mathbb{R} \) with respect to the usual topology. Clearly \( q \in U \setminus \{x\} \), so \( x \in \text{acc}(\mathbb{Q}) \) with respect to \( \mathcal{T} \). Therefore \( \text{acc}(\mathbb{Q}) = \mathbb{R} \), so \( \mathbb{Q} \) is dense in \( \mathbb{R} \) with respect to \( \mathcal{T} \). In particular \( \mathcal{T} \) is separable.

7. Suppose \((x_n)_{n=1}^{\infty} \) has a subsequence \((x_{n_k})_{k=1}^{\infty} \) which converges to \( x \). If \( U \) is a neighbourhood of \( x \), there exists \( N \in \mathbb{N} \) such that \( x_{n_k} \in U \) for all \( k \in \mathbb{N} \) with \( k \geq N \), hence for infinitely many \( k \). Therefore \( x \) is a cluster point of \((x_n)_{n=1}^{\infty}\).

Conversely, suppose that \( x \) is a cluster point of \((x_n)_{n=1}^{\infty}\). Since \( X \) is first countable, there exists a nested countable neighbourhood base \( \{U_n\}_{n=1}^{\infty} \) at \( x \). Set \( n_0 := 0 \) and, for each \( k \in \mathbb{N} \), choose \( n_k \in \mathbb{N} \) inductively so that \( n_k > n_{k-1} \) and \( x_{n_k} \in U_k \). This is possible because \( \{m \in \mathbb{N} \mid x_m \in U_k\} \) is infinite and hence \( \{m \in \mathbb{N} \mid x_m \in U_k\} \setminus [1, n_{k-1}] \neq \emptyset \). If \( U \) is a neighbourhood of \( x \) then \( U_N \subseteq U \) for some \( N \in \mathbb{N} \). It follows that \( x_{n_k} \in U_k \subseteq U_N \subseteq U \) for all \( k \in \mathbb{N} \) with \( k \geq N \). This shows that the subsequence \((x_{n_k})_{k=1}^{\infty} \) of \((x_n)_{n=1}^{\infty} \) converges to \( x \).

10. (a) Suppose \( X \) is connected, and let \( A \subseteq X \) be clopen. Then \( A \) and \( A^c \) are disjoint open sets which cover \( X \). Since \( X \) is connected, it follows that \( A = \emptyset \) or \( A^c = \emptyset \). Therefore \( A \) and \( X \) are the only clopen subsets of \( X \).

Conversely, suppose that \( \emptyset \) and \( X \) are the only clopen subsets of \( X \). If \( U, V \subseteq X \) are disjoint open sets which cover \( X \), then \( U = V^c \) and hence \( U \) is clopen. This implies that \( U \in \{\emptyset, X\} \), so \( U = \emptyset \) or \( V = \emptyset \). Therefore \( X \) is not disconnected, i.e. it is connected.

(b) Define \( E := \bigcup_{\alpha \in A} E_\alpha \) and suppose that \( U, V \subseteq E \) are non-empty open sets (relative to \( E \)) which cover \( E \). Choose \( x \in \bigcap_{\alpha \in A} E_\alpha \), and without loss of generality assume \( x \in U \). Also choose \( y \in V \), so that \( y \in E_\alpha \) for some \( \alpha \in A \). Now \( x \in U \cap E_\alpha \) and \( y \in V \cap E_\alpha \), so \( U \cap E_\alpha \) and \( V \cap E_\alpha \) are non-empty open sets (relative to \( E_\alpha \)) which cover \( E_\alpha \). Since \( E_\alpha \) is connected, these sets cannot be disjoint. Therefore \( U \cap V \neq \emptyset \), which shows that \( E \) is connected.

(c) Let \( U, V \subseteq \overline{A} \) be disjoint open sets (relative to \( \overline{A} \)) which cover \( \overline{A} \). Then \( U = U' \cap \overline{A} \) and \( V = V' \cap \overline{A} \) for some open sets \( U', V' \subseteq X \) which cover \( A \) such that \( U' \cap V' \cap \overline{A} = \emptyset \). It follows that \( U' \cap \overline{A} \) and \( V' \cap \overline{A} \) are disjoint open sets (relative to \( A \)) which cover \( \overline{A} \). Since \( A \) is connected, \( U' \cap \overline{A} = \emptyset \) without loss of generality. This implies that \( U' \cap \text{acc}(A) = \emptyset \), because if \( x \in U' \cap \text{acc}(A) \) then \( (U' \setminus \{x\}) \cap A \neq \emptyset \), which is impossible. Therefore \( U' \cap \overline{A} = \emptyset \), because \( \overline{A} = A \cup \text{acc}(A) \), and hence \( \overline{A} \) is connected.
13. Clearly $U \cap \overline{A} \subseteq U$. Conversely, note that $A$ is contained in the closed set $U^c \cup (U \cap \overline{A})$, and hence $U^c \cup (U \cap \overline{A}) = X$. This implies that $U \subseteq U \cap \overline{A}$, so $U \subseteq U \cap \overline{A}$. Therefore $\overline{U} = U \cap \overline{A}$.

14. Suppose that $f$ is continuous and let $A \subseteq X$. Then $f^{-1}(\overline{f(A)})$ is closed (as $\overline{f(A)}$ is closed) and $A \subseteq f^{-1}(\overline{f(A)})$ because $f(A) \subseteq \overline{f(A)}$. It follows that $\overline{A} \subseteq f^{-1}(\overline{f(A)})$, which implies that $f(\overline{A}) \subseteq \overline{f(A)}$.

Now suppose that $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$. If $B \subseteq Y$, then $f^{-1}(B) \subseteq X$ and hence $f(f^{-1}(B)) \subseteq \overline{f(f^{-1}(B))}$. Since $f(f^{-1}(B)) \subseteq B$, this implies that $f(f^{-1}(B)) = B$, or equivalently $f^{-1}(B) \subseteq f^{-1}(B)$.

Finally, suppose that $f^{-1}(B) \subseteq f^{-1}(B)$ for all $B \subseteq Y$. If $C \subseteq Y$ is closed, then $f^{-1}(C) \subseteq f^{-1}(\overline{C}) = f^{-1}(C)$. This implies that $f^{-1}(C) = f^{-1}(C)$, so $C$ is closed and hence $f$ is continuous.

16. (a) Consider the following diagrams:

![Diagram](image)

There exist unique maps $\Delta : Y \to Y \times Y$ and $h : X \to Y \times Y$ which make the resulting diagrams commute, and they are both continuous by Proposition 4.11. Note that $\Delta(Y) = \{(y,y) \mid y \in Y\}$ is closed, because $\Delta(Y)^c$ is open: if $(a,b) \in \Delta(Y)^c$ then $a \neq b$ and there exist disjoint open neighbourhoods $U$ and $V$ of $a$ and $b$ respectively; in particular $(a,b) \in U \times V \subseteq \Delta(Y)^c$. It follows that $\{x \in X \mid f(x) = g(x)\} = h^{-1}(\Delta(Y))$ is closed.

(b) Since $\{x \in X \mid f(x) = g(x)\}$ is closed and contains a dense subset of $X$, it is equal to $X$. Therefore $f = g$.

17. Suppose that, for every $x, y \in X$ with $x \neq y$, there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$. Given $x, y \in X$ with $x \neq y$, choose such a function $f$ and disjoint open sets $U, V \subseteq \mathbb{R}$ such that $f(x) \in U$ and $f(y) \in V$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint members of $\mathcal{F}$ with $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. This shows that $\mathcal{F}$ is Hausdorff.

Now suppose that $\mathcal{F}$ is Hausdorff, and let $x, y \in X$ be distinct. There exists $U \in \mathcal{F}$ such that $x \in U$ and $y \notin U$. Clearly $\emptyset \neq U \neq X$, so there exist $U_1, U_2, \ldots, U_n \subseteq \mathbb{R}$ and $f_1, f_2, \ldots, f_n \in \mathcal{F}$ such that $x \in \bigcap_{k=1}^n f^{-1}_k(U_k) \subseteq U$. It follows that $y \notin f^{-1}_k(U_k)$ for some $k \in \{1, 2, \ldots, n\}$, and hence $f_k(x) \neq f_k(y)$.

22. If $x_0 \in X$, then $0 \leq \rho(f_n(x_0), f_m(x_0)) \leq \sup_{x \in X} \rho(f_n(x), f_m(x))$ for all $m, n \in \mathbb{N}$. Therefore $(f_n(x_0))_{n=1}^\infty$ is a Cauchy sequence in the complete metric space $(Y, \rho)$, which has a limit $f(x_0)$. This defines a map $f : X \to Y$. Given $\varepsilon \in (0, \infty)$ there exists $N \in \mathbb{N}$ such that $\sup_{x \in X} \rho(f_n(x), f_m(x)) < \frac{\varepsilon}{2}$ for all $m, n \in \mathbb{N}$ with $m \geq n \geq N$. If $x \in X$ and $n \in \mathbb{N}$ with $n \geq N$ then $\rho(f_n(x), f(x)) < \frac{\varepsilon}{2}$ for some $m \in \mathbb{N}$ with $m \geq n$, so $\rho(f_n(x), f(x)) < \varepsilon$. This implies that $\sup_{x \in X} \rho(f_n(x), f(x)) \leq \varepsilon$ for all $n \in \mathbb{N}$ with $n \geq N$, so $(\sup_{x \in X} \rho(f_n(x), f(x)))_{n=1}^\infty$ converges to 0.
Suppose $g : X \to Y$ and $(\sup_{x \in X} \rho(f_n(x), g(x)))_{n=1}^{\infty}$ converges to $0$. If $\varepsilon \in (0, \infty)$ and $x \in X$ then $\rho(f_n(x), f(x)) < \frac{\varepsilon}{2}$ and $\rho(f_n(x), g(x)) < \frac{\varepsilon}{2}$ for some $N \in \mathbb{N}$, which implies that $\rho(f(x), g(x)) < \varepsilon$. Therefore $\rho(f(x), g(x)) = 0$, so $f(x) = g(x)$ and hence $f = g$. This shows that $f$ is unique.

Now suppose that $f_n$ is continuous for all $n \in \mathbb{N}$. Let $x_0 \in X$ and suppose that $U \subseteq Y$ is a neighbourhood of $f(x_0)$. There exists $\varepsilon \in (0, \infty)$ such that $B_\varepsilon(f(x_0)) \subseteq U$, and there exists $N \in \mathbb{N}$ such that $\sup_{x \in X} \rho(f_n(x), f(x)) < \frac{\varepsilon}{3}$. Since $f_N$ is continuous, $V := f_N^{-1}(B_{\varepsilon/3}(f_N(x_0)))$ is an open neighbourhood of $x_0$. If $x \in V$ then

$$\rho(f(x), f(x_0)) \leq \rho(f(x), f_N(x)) + \rho(f_N(x), f_N(x_0)) + \rho(f_N(x_0), f(x_0)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

and hence $f(V) \subseteq B_{\varepsilon}(f(x_0)) \subseteq U$. This implies that $f^{-1}(U)$ is a neighbourhood of $x_0$, so $f$ is continuous.

23. Let $A \subseteq \mathbb{R}$ be closed $a \in \mathbb{R}$, $b \in (a, \infty)$ and $f \in C(A, [a, b])$. We aim to extend $f$ to some $F \in C(\mathbb{R}, [a, b])$. Since $A^c$ is open, it is a disjoint union of open intervals $\{(a_n, b_n)\}_{n=1}^{\infty}$. At most two of these are unbounded, and on these we let $F$ be constant, equal to the value of $f$ at the finite endpoint. On bounded intervals $(a_n, b_n)$, we let $F$ be the linear function from $(a_n, f(a_n))$ to $(b_n, f(b_n))$. By construction $F$ has the same image as $f$, and is continuous on $A^c$.

Given $x \in A$, we will show that $F$ is right continuous at $x$; the proof of left continuity is similar. For each $\varepsilon \in (0, \infty)$, there exists $\delta \in (0, \infty)$ such that $|f(x) - f(y)| < \varepsilon$ for all $y \in (x, x + \delta) \cap A$. If $(x, x + \delta) \subseteq A$ then we are done. Otherwise $(x, x + \delta)$ meets $(a_n, b_n)$ for some $n \in \mathbb{N}$; this implies that $a_n = x$ or $a_n \in (x, x + \delta)$. If $a_n = x$ we are done, since $F$ is continuous on $[a_n, b_n]$. Otherwise $|F(x) - F(y)| < \varepsilon$ for all $y \in (a_n, b_n)$: if $y \in A$ we know this already, and if $y \in (a_m, b_m)$ for some $m \in \mathbb{N}$ then $F(y)$ is between $f(a_m)$ and $f(b_m)$, which are both within $\varepsilon$ of $f(x) = F(x)$ because $x \leq a_m < b_m < a_n < x + \delta$ and $a_m, b_m \in A$. Therefore $F$ is right continuous at $x$, and we are done.

24. If $X$ is normal, then by Urysohn’s lemma and the Tietze extension theorem, it satisfies the conclusions of Urysohn’s lemma and the Tietze extension theorem. Conversely, if $X$ satisfies the conclusion of Urysohn’s lemma, we claim that $X$ is normal. To this end, let $A$ and $B$ be disjoint closed subsets of $X$. There exists $f \in C(X, [0, 1])$ such that $f|_A = 0$ and $f|_B = 1$. Note that $f^{-1}((\infty, \frac{1}{2f})$ and $f^{-1}((\frac{3}{2f}, \infty))$ are disjoint open sets which contain $A$ and $B$ respectively. Therefore $X$ is normal. Finally, suppose that $X$ satisfies the conclusion of the Tietze extension theorem. We need to show that $X$ is normal; by the above it suffices to show that $X$ satisfies the conclusion of Urysohn’s lemma. To this end, let $A$ and $B$ be disjoint closed subsets of $X$. Note that $A \cup B$ is closed, and that $A$ and $B$ are open in $A \cup B$. Therefore $\chi_B \in C(A \cup B, [0, 1])$, and there exists $f \in C(X, [0, 1])$ extending $\chi_B$, as required.

26. (a) As is often the case, it is easier to talk about disconnected spaces. We need to show that, if $f(X)$ is disconnected, then $X$ is disconnected. Since $f(X)$ is disconnected, it has a nonempty proper subset $A$ which is both open and closed (clopen). Note that $A = U \cap f(X)$ for some open set $U \subseteq Y$, and $A = C \cap f(X)$ for some closed set $C \subseteq Y$, by definition of the relative topology on $f(X)$. Therefore $f^{-1}(A) = f^{-1}(U) = f^{-1}(C)$ is a nonempty proper clopen subset of $X$. This implies that $X$ is disconnected.

(b) Again we prove the contrapositive: if $X$ is disconnected, then it is not arcwise connected (usually this is called path connected). Let $A$ be a nonempty proper clopen subset of $X$. There exists $x_0 \in A$ and $x_1 \in A^c$, but there is no path joining these points: if there was such a path, say $f$, then $f([0, 1]) \cap A$ is a nonempty proper clopen subset of $f([0, 1])$, which is impossible by part (a).

(c) By part (a) the spaces $X^+ := X \cap ((0, \infty) \times \mathbb{R})$ and $X^- := X \cap ((-\infty, 0) \times \mathbb{R})$ are connected. If $A$ is a clopen subset of $X$, we need to show that $A = \emptyset$ or $A = X$. Without loss of generality $(0, 0) \in A$ (otherwise, take $A^c$). Clearly, there is a sequence in $X^+$ which converges to $(0, 0)$. Since $A$ is a neighbourhood of $(0, 0)$, it follows that
A ∩ X^+ is a nonempty clopen subset of X^+. This implies that A ∩ X^+ = X^+. Similarly, A \cap X^- = X^-\right. Therefore \ A = X, which shows that X is connected.

If X is path connected, there should be a path f from (0, 0) to (\frac{1}{n}, 0). Define t_0 := \sup f^{-1}(\{(0, 0)\}) and note that f(t_0) = (0, 0) because f^{-1}(\{(0, 0)\}) is compact. Since f is continuous, there exists t_1 \in (t_0, 1) such that f(t) \in B_{\frac{1}{2}}(0, 0) for all t \in [t_0, t_1]. Note that f(t_1) \in Y, so \pi_1(f(t_1)) \neq 0, where \pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R} is the projection onto the x-axis. In particular, there is a point (x, 1) of Y such that x is between \pi_1(f(t_1)) and 0. By part (a) \pi_1(f([t_0, t_1])) is connected, so it contains x (if not, then \ (-\infty, x] \cap \pi_1(f([t_0, t_1])) is a nonempty proper clopen subset) and hence f([t_0, t_1]) contains (x, 1). This contradicts the fact that f([t_0, t_1]) \subseteq B_{\frac{1}{2}}(0, 0).

27. If X = \emptyset then X is connected. Otherwise choose x \in X and let C \subseteq X be the connected component of x. Suppose that C \neq X. Then there exists y \in C^c, and C is closed (by the previous homework), so there exist \alpha_1, \alpha_2, \ldots, \alpha_n \in A and \ U_{\alpha_1} \subseteq X_{\alpha_1}, U_{\alpha_2} \subseteq X_{\alpha_2}, \ldots, U_{\alpha_n} \subseteq X_{\alpha_n} open such that y \in \cap_{k=1}^n \pi_{\alpha_k}^{-1}(U_{\alpha_k}) \subseteq C^c. Define \ z_k \in X by

\[ \pi_{\alpha}(z_k) := \begin{cases} \pi_{\alpha}(y), & \alpha \in \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \\ \pi_{\alpha}(x), & \alpha \notin \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \end{cases} \]

for each k \in \{0, 1, \ldots, n\}, so that z_0 = x and z_n \in \cap_{k=1}^n \pi_{\alpha_k}^{-1}(U_{\alpha_k}). Given k \in \{1, 2, \ldots, n\} define \ i_k : X_{\alpha_k} \rightarrow X by

\[ \pi_{\alpha}(i_k(p)) := \begin{cases} p, & \alpha = \alpha_k \\ \pi_{\alpha}(z_k), & \alpha \neq \alpha_k \end{cases} \]

so that \pi_{\alpha} \circ i_k is continuous for all \alpha \in A and \ i_k is continuous. In particular \ i_k(X_{\alpha_k}) is connected, and it contains both \ z_k and \ z_{k-1}. Since z_0 = x this implies that \ i_1(X_{\alpha_1}) \subseteq C, and by induction it follows that z_n \in i_n(X_{\alpha_n}) \subseteq C. But z_n \in C^c, which is a contradiction. Therefore C = X, so X is connected.

32. Suppose X is Hausdorff and let \langle x_\alpha \rangle_{\alpha \in A} be a net in X which converges to x \in X. If y \in X and x \neq y, there exist disjoint open sets U, V \subseteq X such that x \in U and y \in V. There exists \beta \in A such that x_\alpha \in U for all \alpha \in A with \alpha \gtrsim \beta. If \gamma \in A then there exists \delta \in A such that \delta \gtrsim \beta and \delta \gtrsim \gamma, which implies that \langle x_\alpha \rangle_{\alpha \in A} is not eventually in V because x_\delta \notin V. Therefore x is the only limit point of \langle x_\alpha \rangle_{\alpha \in A}.

Conversely, suppose that X is not Hausdorff, and choose \ x, y \in X distinct with no disjoint open neighbourhoods. For each pair \langle N_x, N_y \rangle \in X \times X we may choose \langle x_{N_x}, N_y \rangle \in X \times N_y, because if \ N_x and N_y were disjoint, they would contain disjoint open neighbourhoods of x and y respectively. If \ N_x \in X, choose \ N_y \in N_y and note that \langle x_{O_x}, O_y \rangle \in O_x \cap O_y \subseteq N_x for all \langle O_x, O_y \rangle \in X \times N_y such that \langle O_x, O_y \rangle \gtrsim \langle N_x, N_y \rangle. This implies that \langle x_{N_x}, N_y \rangle_{\langle N_x, N_y \rangle \in X \times N_y} converges to x, and a similar argument shows that it also converges to y.

34. If \langle x_\alpha \rangle_{\alpha \in A} is a net in X which converges to x \in X, and f \in \mathcal{F}, then f is continuous so \langle f(x_\alpha) \rangle_{\alpha \in A} converges to f(x).

Conversely, let \langle x_\alpha \rangle_{\alpha \in A} be a net in X and suppose there exists x \in X such that \langle f(x_\alpha) \rangle_{\alpha \in A} converges to f(x) for all f \in \mathcal{F}. Given a neighbourhood U \subseteq X of x, there exist f_1, f_2, \ldots, f_n \in \mathcal{F} and U_1 \subseteq f_1(X), U_2 \subseteq f_2(X), \ldots, U_n \subseteq f_n(X) open such that x \in \cap_{k=1}^n f_k^{-1}(U_k) \subseteq U. For each k \in \{1, 2, \ldots, n\} there exists \beta_k \in A such that f_k(x_\alpha) \in U_k for all \alpha \in A with \alpha \gtrsim \beta_k. By induction there exists an upper bound \beta for \{\beta_1, \beta_2, \ldots, \beta_n\}, and it follows that \ x_\alpha \in \cap_{k=1}^n f_k^{-1}(U_k) \subseteq U \ \text{for all} \ \alpha \in A \ \text{with} \ \alpha \gtrsim \beta. \ \text{Therefore} \ \langle x_\alpha \rangle_{\alpha \in A} \converges to x.

36. Suppose there exists a topology \mathcal{F} on X in which convergence corresponds to pointwise almost everywhere convergence. For each n \in \mathbb{N} \cup \{0\} and k \in \{0, 1, \ldots, 2^n - 1\} define f_{2^n+k} := \chi_{[2^{-n}k,2^{-n}(k+1)]}. Since it takes on each of the values 0
and 1 infinitely many times, \((f_n(x))_{n=1}^{\infty}\) fails to converge for all \(x \in [0, 1]\). In particular, \((f_n)_{n=1}^{\infty}\) does not converge to 0 in \((X, \mathcal{T})\). Hence there exists \(U \in \mathcal{T}\) such that \(0 \in U\) with the property that for each \(N \in \mathbb{N}\) there exists \(n \in \mathbb{N}\) such that \(n \geq N\) and \(f_n \notin U\). Define \(n_0 := 0\), and for each \(k \in \mathbb{N}\) choose \(n_k \in \mathbb{N}\) so that \(n_k > n_{k-1}\) and \(f_{n_k} \notin U\). Clearly \((f_{n_k})_{k=1}^{\infty}\) converges to 0 in \(L^1\) (as does \((f_n)_{n=1}^{\infty}\)), so it has a subsequence \((f_{n_k})_{j=1}^{\infty}\) which converges to 0 pointwise almost everywhere. This is impossible because \(f_{n_kj} \notin U\) for all \(j \in \mathbb{N}\).

38. Suppose that \(\mathcal{T} \subseteq \mathcal{T}'\). If \(x, y \in X\) and \(x \neq y\), there exist disjoint sets \(U, V \in \mathcal{T} \subseteq \mathcal{T}'\) such that \(x \in U\) and \(y \in V\). Therefore \((X, \mathcal{T}')\) is Hausdorff. Suppose that \((X, \mathcal{T}')\) is compact, and let \(V \in \mathcal{T}'\). Then \(V^c\) is closed in \(\mathcal{T}'\), hence compact relative to \(\mathcal{T}'\). The identity map from \((X, \mathcal{T}')\) to \((X, \mathcal{T})\) is continuous, so \(V^c\) is compact relative to \(\mathcal{T}\). Since \(\mathcal{T}\) is Hausdorff \(V^c\) is closed in \(\mathcal{T}\), so \(V \in \mathcal{T}\) and hence \(\mathcal{T} = \mathcal{T}'\). In particular, if \(\mathcal{T} \subset \mathcal{T}'\) then \((X, \mathcal{T}')\) is not compact.

Conversely, suppose that \(\mathcal{T}' \subset \mathcal{T}\). The identity map from \((X, \mathcal{T})\) to \((X, \mathcal{T}')\) is continuous, so \((X, \mathcal{T}')\) is compact. Therefore \((X, \mathcal{T}')\) is not Hausdorff, by the previous argument.

39. Suppose a space \(X\) has a countable open cover \(\{U_n\}_{n=1}^{\infty}\) with no finite subcover. For each \(n \in \mathbb{N}\) there exists \(x_n \in (\bigcup_{k=1}^{n} U_k)^c\). Let \(\langle x_{n_k} \rangle_{k=1}^{\infty}\) be a subsequence of \(\langle x_k \rangle_{k=1}^{\infty}\) and let \(x_0 \in X\). Then \(x_0 \in U_m\) for some \(m \in \mathbb{N}\), and \(x_{n_k} \notin U_m\) for sufficiently large \(k \in \mathbb{N}\). This implies that \(\langle x_{n_k} \rangle_{k=1}^{\infty}\) does not converge, so \(X\) is not sequentially compact.

40. Let \(\langle x_n \rangle_{n=1}^{\infty}\) be a sequence in \(X\), and for each \(n \in \mathbb{N}\) define \(E_n := \{x_k\}_{k=n}^{\infty}\). If \(\bigcap_{n=1}^{\infty} E_n = \emptyset\), then \(\langle (E_n)^c \rangle_{n=1}^{\infty}\) is a countable open cover of \(X\), which has a finite subcover \((\langle E_{n_1} \rangle^c, \langle E_{n_2} \rangle^c, \ldots, \langle E_{n_m} \rangle^c)\). It follows that

\[
\bigcap_{k=1}^{m} E_{n_k} \subseteq \bigcap_{k=1}^{m} (E_{n_k})^c = (\bigcup_{k=1}^{m} (E_{n_k})^c)^c = X^c = \emptyset,
\]

which is a contradiction because \(x_N \in \bigcap_{k=1}^{N} E_{n_k}\) for \(N := \max\{n_k\}_{k=1}^{m}\). Hence there exists \(x_0 \in \bigcap_{n=1}^{\infty} E_n\). If \(U \subseteq X\) is a neighbourhood of \(x_0\) and \(n \in \mathbb{N}\) then \(U \setminus \{x_0\}\) meets \(E_n\), so there exists \(k \in \mathbb{N}\) such that \(k \geq n\) and \(x_k \in U\). This implies that \(x_0\) is a cluster point of \(\langle x_n \rangle_{n=1}^{\infty}\). By exercise 7 (see above), it follows that \(\langle x_n \rangle_{n=1}^{\infty}\) has a convergent subsequence provided that \(X\) is first countable.

43. Let \(\langle a_{n_k} \rangle_{k=1}^{\infty}\) be a subsequence of \(\langle a_k \rangle_{k=1}^{\infty}\), and define \(x := \sum_{k=1}^{\infty} (1 + (-1)^k) 2^{-(nk+1)}\). Then \(x \in [0, 1)\) (in fact \(x \in [0, \frac{1}{4})\)) and \(a_{n_k}(x) = \frac{1}{2} (1 + (-1)^k)\) for all \(k \in \mathbb{N}\). The sequence \(\langle a_{n_k}(x) \rangle_{k=1}^{\infty} = (0, 1, 0, 1, \ldots)\) does not converge, so \(\langle a_{n_k} \rangle_{k=1}^{\infty}\) does not converge in \([0, 1])^{[0,1]}\) and hence \(\langle a_{n_k} \rangle_{k=1}^{\infty}\) has no convergent subsequence.

44. If \(\{U_n\}_{n=1}^{\infty}\) is a (countable) open cover of \(f(X)\), then \(\{f^{-1}(U_n)\}_{n=1}^{\infty}\) is an open cover of \(X\), which has a finite subcover, say \(\{f^{-1}(U_n)\}_{n=1}^{N}\). It follows that \(\{U_n\}_{n=1}^{N}\) covers \(f(X)\), which shows that \(f(X)\) is countably compact.

45. Suppose \(X\) is countably compact. If \(f \in C(X)\), then \(f(X)\) is countably compact by Exercise 44. Since \(C\) is first countable, it follows that \(f(X)\) is sequentially compact by Exercise 40, and hence bounded by Heine-Borel.
49. (a) If \( x \in E \), then \( x \) has a compact neighbourhood \( N \subseteq E \). This contains an open neighbourhood \( U \subseteq N \) of \( x \), and \( U = U \cap E \) is relatively open, so \( N \) is a neighbourhood of \( x \) in the relative topology. It is actually a compact neighbourhood, because the topologies on \( N \) relative to \( E \) and \( X \) are the same. Therefore \( E \) is locally compact.
(b) Let \( x \in E \) and choose a compact neighbourhood \( N \subseteq E \) of \( x \) in the relative topology. As \( N \) is a compact subspace of the Hausdorff space \( X \), it is closed. Let \( U \subseteq N \) be a relatively open neighbourhood of \( x \), so that \( U = V \cap E \) for some open \( V \subseteq X \). By Exercise 13, \( x \in V \subseteq \overline{V} \cap E \subseteq N \subseteq E \), which shows that \( E \) is open.
(c) Suppose \( E \) is locally compact in the relative topology. Since \( \overline{E} \subseteq X \) is closed, it is compact and \( \overline{E} \) is a compact Hausdorff space. The relative topologies on \( E \) inherited from \( \overline{E} \) and \( X \) are the same, so \( E \) is locally compact in the relative topology inherited from \( \overline{E} \). If \( C \subseteq \overline{E} \) is relatively closed and \( E \subseteq C \), then \( C = D \cap \overline{E} \) for some closed \( D \subseteq X \), so \( C \) is closed and hence \( C = \overline{E} \). This implies that \( E \) is dense in \( \overline{E} \), so \( E \) is relatively open in \( \overline{E} \) by part (b) with \( X \) replaced by \( \overline{E} \).

Conversely, suppose that \( E \) is relatively open in \( \overline{E} \). Again, note that \( \overline{E} \) is a compact Hausdorff space, in which case \( E \) is locally compact in the relative topology inherited from \( \overline{E} \), by part (a). This is the same as the relative topology inherited from \( X \), so \( E \) is also locally compact in that topology.

51. Suppose \( \phi \) is proper, and extend it to a map \( X^* \to Y^* \). Let \( U \subseteq Y^* \) be open. If \( \infty_Y \notin U \) then \( \phi^{-1}(U) \) is open, because \( \phi|_X \) is continuous. Otherwise \( U^c \subseteq Y \) is compact, so \( \phi^{-1}(U^c) \) is compact and hence \( \phi^{-1}(U) = \phi^{-1}(U^c)^c \) is open (as \( X^* \) is Hausdorff, or because \( \infty_X \in \phi^{-1}(U) \)). Therefore \( \phi \) is continuous.

Conversely, suppose that \( \phi \) extends continuously to a map \( X^* \to Y^* \). If \( K \subseteq Y \) is compact, then \( K^c \) is open in \( Y^* \), so \( \phi^{-1}(K^c) \) is open in \( X^* \). Since \( \infty_X \in \phi^{-1}(K^c) \), it follows that \( \phi^{-1}(K) = \phi^{-1}(K^c)^c \) is compact. Thus \( \phi \) is proper.

54. (a) If \( K \subseteq \mathbb{Q} \) is a neighbourhood of 0, then \( (-\varepsilon, \varepsilon) \cap \mathbb{Q} \subseteq K \) for some \( \varepsilon \in (0, \infty) \cap \mathbb{Q} \). Clearly \( \frac{\varepsilon}{\sqrt{2}} \in \overline{K} \setminus K \), so \( K \) is not a closed subset of \( \mathbb{R} \); in particular it is not compact. Therefore 0 has no compact neighbourhood in \( \mathbb{Q} \).
(b) Define \( K := \{0\} \cup \{\frac{1}{n}\}_{n=1}^\infty \), and note that \( K^c = \bigcup_{n=1}^\infty \left(\frac{1}{n+1}, \frac{1}{n}\right) \cup (-\infty,0) \cup (1,\infty) \) is compact and bounded, hence compact. For each \( n \in \mathbb{N} \) let \( f_n : \mathbb{Q} \to \mathbb{C} \) be the indicator function of \( \{\frac{1}{n}\} \). The sequence \( \langle f_n \rangle_{n=1}^\infty \) converges to 0 pointwise, but not uniformly on \( K \).

56. (a) If \( t \in (0, \infty) \) then \( \Phi'(t) = (t+1)^{-2} > 0 \), so \( \Phi \) is strictly increasing on \( [0, \infty) \) by the mean value theorem. Since \( \Phi(t) < 1 \) for all \( t \in [0, \infty) \), it follows that \( \Phi \) is strictly increasing. If \( s,t \in [0, \infty) \) then

\[
\Phi(t+s) = \frac{t+s}{1+t+s} = \frac{t}{1+t+s} + \frac{s}{1+t+s} \leq \frac{t}{1+t} + \frac{s}{1+s} = \Phi(s) + \Phi(t),
\]

and the same clearly holds if \( s = \infty \) or \( t = \infty \).
(b) If \( y \in Y \) then \( \Phi(\rho(y,y)) = \Phi(0) = 0 \). Conversely, if \( x,y \in Y \) and \( \Phi(\rho(x,y)) = 0 \) then \( \rho(x,y) = 0 \) (as \( \Phi \) is injective) and hence \( x = y \). Clearly \( \Phi \circ \rho \) is nonnegative and bounded. If \( x,y,z \in Y \) then \( \Phi(\rho(x,y)) = \Phi(\rho(y,x)) \) and \( \Phi(\rho(x,z)) \leq \Phi(\rho(x,y) + \rho(y,z)) \leq \Phi(\rho(x,y)) + \Phi(\rho(y,z)) \). Therefore \( \Phi \circ \rho \) is a metric.

If \( y \in Y \) and \( r \in (0, \infty) \) then

\[
B_r \Phi^1(y) = \{ x \in Y \mid \rho(x,y) < r \} = \{ x \in Y \mid \Phi(\rho(x,y)) < \Phi(r) \} = B_{\Phi(r)} \Phi^1(y).
\]

Moreover, if \( r < 1 \) then \( r = \Phi(\frac{r}{1-r}) \) and hence

\[
B_r \Phi^0 (y) = \{ x \in Y \mid \Phi(\rho(x,y)) < \Phi(\frac{r}{1-r}) \} = \{ x \in Y \mid \rho(x,y) < \frac{r}{1-r} \} = B_{\frac{r}{1-r}}(y).
\]

Otherwise \( B_r \Phi^0 (y) = Y \). This shows that \( \rho \) and \( \Phi \circ \rho \) define the same topologies on \( Y \).
(c) The proof is the same as part (b), except that

$$B_1(g) = \{ f \in \mathbb{C}^X \mid \rho(f, g) < 1 \} = \{ f \in \mathbb{C}^X \mid \sup_{x \in X} |f(x) - g(x)| < \infty \}$$

for all $g \in \mathbb{C}^X$. These sets are still open in the topology of uniform convergence, as required.

(d) It is routine to check that $\rho$ is a metric. Let $r \in (0, \infty)$ and $f \in \mathbb{C}^X$. For each $g \in B_r(f)$ there exists $\varepsilon \in (0, \infty)$ such that $B_\varepsilon(g) \subseteq B_r(f)$. Choose $m, N \in \mathbb{N}$ so that $\Phi(m^{-1}) < \frac{\varepsilon}{2}$ and $\sum_{n=N}^{\infty} 2^{-n} < \frac{\varepsilon}{2}$. Note that

$$\{ h \in \mathbb{C}^X \mid \sup_{x \in \mathcal{U}_N} |h(x) - g(x)| < m^{-1} \} \subseteq \left\{ h \in \mathbb{C}^X \mid \sum_{n=1}^{N-1} 2^{-n} \Phi \left( \sup_{x \in \mathcal{U}_n} |h(x) - g(x)| \right) < \frac{\varepsilon}{2} \right\} \subseteq B_\varepsilon(g).$$

This shows that $B_r(f)$ is open in the topology of uniform convergence on compact sets. Conversely, let $f \in \mathbb{C}^X$ and $m, n \in \mathbb{N}$. For each $g \in \mathbb{C}^X$ with $\sup_{x \in \mathcal{U}_n} |g(x) - f(x)| < m^{-1}$, there exists $\varepsilon \in (0, \infty)$ such that

$$\{ h \in \mathbb{C}^X \mid \sup_{x \in \mathcal{U}_n} |h(x) - g(x)| < \varepsilon \} \subseteq \{ h \in \mathbb{C}^X \mid \sup_{x \in \mathcal{U}_n} |h(x) - f(x)| < m^{-1} \}.$$  

It is clear that $B_{2^{-n}\Phi(\varepsilon)}(g)$ is contained in the former set. This shows that the latter set is open in the metric topology, so the two topologies are equivalent.

57. (a) Let $\{ U_\alpha \}_{\alpha \in A}$ be an open cover of $X$, and choose a sequence $\langle E_n \rangle_{n=1}^{\infty}$ of precompact open subsets of $X$ such that $\overline{E}_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} E_n = X$. Set $E_0 := \emptyset$ and for each $n \in \mathbb{N}$ define $U_n := \{ U_\alpha \cap (E_{n+2} \setminus \overline{E}_{n-1}) \}_{\alpha \in A}$, so that $\mathcal{U}_n$ is an open cover of the compact set $\overline{E}_{n+1} \setminus E_n$ (or $\overline{E}_2$ in the case $n = 1$), having a finite subcover $\mathcal{V}_n$. Write $\bigcup_{n=1}^{\infty} \mathcal{V}_n = \{ V_\alpha \}_{n=1}^{\infty}$, and note that this cover is a refinement of $\{ U_\alpha \}_{\alpha \in A}$. We claim it is locally finite. Indeed, if $x \in X$ then $x \in E_{n+2} \setminus \overline{E}_{n-1}$ for some $n \in \mathbb{N}$, which is an open neighbourhood of $x$ that meets no members of $\mathcal{V}_m$ for all $m \in \mathbb{N}$ with $|m - n| \geq 3$. If $n \in \mathbb{N}$ then $\mathcal{V}_n$ is a closed subset of $\overline{E}_{n+2}$ for some $m \in \mathbb{N}$, and is therefore compact.

For each $x \in X$ there exists $n \in \mathbb{N}$ such that $x \in V_n$, and $x$ has a compact neighbourhood $N_x \subseteq V_n$. Given $n \in \mathbb{N}$, there is a finite set $Y_n \subseteq X$ such that $\overline{V}_n \subseteq \cup_{y \in Y_n} N^\circ_y$. From above it is clear that $V_n$ meets only finitely many members of $\{ V_m \}_{m=1}^{\infty}$, say $\{ V_m \}_{m \in J_n}$, for some finite index set $J_n \subseteq \mathbb{N}$. Define $Z_n := \{ y \in \cup_{m \in J_n} V_m \mid N_y \subseteq V_n \}$ and $W_n := \cup_{y \in Z_n} N^\circ_y$. Since $Z_n$ is finite $\overline{W}_n \subseteq \cup_{y \in Z_n} N_y \subseteq V_n$. Moreover, if $x \in X$ then there exists $n \in \mathbb{N}$ such that $x \in V_n \subseteq \cup_{y \in Y_n} N^\circ_y$, so $x \in N^\circ_y$ for some $y \in Y_n$. By definition $N_y \subseteq V_m$ for some $m \in \mathbb{N}$, and $x \in V_n \cap V_m$ so $n \in J_m$, implying that $y \in Z_m$ and hence $x \in W_m$. This shows that $\{ W_n \}_{n=1}^{\infty}$ is an open cover of $X$. Since it is a refinement of $\{ V_n \}_{n=1}^{\infty}$, it is a locally finite refinement of $\{ U_\alpha \}_{\alpha \in A}$. Note also that $W_n$ is precompact for each $n \in \mathbb{N}$ (as $\overline{W}_n \subseteq \overline{V}_n$).

(b) For each $n \in \mathbb{N}$, by Urysohn’s lemma there exists $f_n \in C_c(X, [0,1])$ such that $f_n(\overline{W}_n) = \{1\}$ and $f_n(V_n^\circ) = 0$, where $\{ W_n \}_{n=1}^{\infty}$ and $\{ V_n \}_{n=1}^{\infty}$ are the covers constructed in part (a). Since $\{ V_n \}_{n=1}^{\infty}$ is locally finite, each $x \in X$ has an open neighbourhood on which $f := \sum_{n=1}^{\infty} f_n$ is well-defined and continuous. Note that $f \geq 1$ because $\{ W_n \}_{n=1}^{\infty}$ covers $X$, and in particular $g_n := f_n/f$ is a well-defined member of $C_c(X, [0,1])$ for each $n \in \mathbb{N}$. It is clear that $\{ g_n \}_{n=1}^{\infty}$ is a partition of unity subordinate to $\mathcal{U}$.

58. Let $K \subseteq \prod_{\alpha \in A} X_\alpha$ be closed, and suppose there exists $x \in K^\circ$. Then there exists a finite subset $B \subseteq A$ and open sets $U_\beta \subseteq X_\beta$ for each $\beta \in B$ such that $x \in \cap_{\beta \in B} \overline{U}_\beta \subseteq K$. Since $B$ is finite, $X_\alpha$ is noncompact for some $\alpha \in A \setminus B$. Hence there exists an open cover $\{ V_\gamma \}_{\gamma \in \Gamma}$ of $X_\alpha$ which has no finite subcover. If $\{ \pi_{\alpha}^{-1}(V_\delta) \}_{\delta \in \Delta}$ covers $K$, for some $\Delta \subseteq \Gamma$, then $\{ V_\delta \}_{\delta \in \Delta}$ covers $X_\alpha$. Indeed, if $y_\alpha \in X_\alpha$ then the point $y \in \prod_{\alpha \in A} X_\alpha$ defined by $\pi_\alpha(y) = y_\alpha$
and $\pi_\beta(y) = \pi_\beta(x)$ for all $\beta \in \mathcal{A} \setminus \{\alpha\}$ lies in $K$, so $y \in \pi_\alpha^{-1}(V_\delta)$ and hence $y_\alpha \in V_\delta$ for some $\delta \in \Delta$. This implies that $\{\pi_\alpha^{-1}(V_\gamma)\}_{\gamma \in \Gamma}$ is an open cover of $K$ with no finite subcover. Therefore $K$ is not compact, which shows that every closed compact subset of $\prod_{\alpha \in \mathcal{A}} X_\alpha$ has empty interior.

59. Let $X$ and $Y$ be locally compact spaces. If $(x, y) \in X \times Y$, then $x$ and $y$ have compact neighbourhoods $K \subseteq X$ and $L \subseteq Y$. By Tychonoff’s theorem $K \times L$ is a compact subset of $X \times Y$, and it is a neighbourhood of $(x, y)$ (by the definition of the product topology). Therefore $X \times Y$ is locally compact; by induction a finite product of locally compact spaces is locally compact.

60. Let $\{X_n\}_{n=1}^\infty$ be a collection of sequentially compact spaces, and let $\langle x_n \rangle_{n=1}^\infty$ be a sequence in $\prod_{n=1}^\infty X_n$. Starting with $\langle x_0 \rangle_{n=1}^\infty := \langle x_n \rangle_{n=1}^\infty$, define for each $m \in \mathbb{N}$ a subsequence $\langle x_m \rangle_{n=1}^\infty$ of $\langle x_{m-1} \rangle_{n=1}^\infty$ such that $\langle \pi_m(x_m) \rangle_{n=1}^\infty$ converges to $x_m \in X_m$. Define $x \in \prod_{n=1}^\infty X_n$ by $\pi_n(x) = x_n$ for all $n \in \mathbb{N}$, and let $U \subseteq \prod_{n=1}^\infty X_n$ be a neighbourhood of $x$. There exists $N \in \mathbb{N}$ and open subsets $U_n \subseteq X_n$ for each $n \in \{1, 2, \ldots, N\}$ such that $x \in \bigcap_{n=1}^N \pi_n^{-1}(U_n) \subseteq U$. Given $m \in \{1, 2, \ldots, N\}$, $\langle x_m \rangle_{n=1}^\infty$ is a subsequence of $\langle x_{m-1} \rangle_{n=1}^\infty$, so there exists $N_m \in \mathbb{N}$ such that $\pi_m(x_n) = x_m$ for all $n \in \mathbb{N}$ with $n \geq N_m$. If $n \in \mathbb{N}$ and $n \geq \max\{N_m\}_{m=1}^N$, then $\pi_m(x_n) = x_m$ for all $m \in \{1, 2, \ldots, N\}$, and hence $x_n \in U$. This shows that $\langle x_n \rangle_{n=1}^\infty$ (a subsequence of $\langle x_{n-1} \rangle_{n=1}^\infty$) converges to $x$. Therefore $\prod_{n=1}^\infty X_n$ is sequentially compact.

63. If $f = 0$ then $Tf = 0 \in C([0, 1])$. Otherwise $\|f\|_u > 0$. Let $\varepsilon \in (0, \infty)$ and note that $K$ is uniformly continuous (because $[0, 1]^2$ is compact). Hence there exists $\delta \in (0, \infty)$ such that $\|K(z_1) - K(z_2)\| < \varepsilon/\|f\|_u$ for all $z_1, z_2 \in [0, 1]^2$ with $|z_1 - z_2| < \delta$. It follows that

$$|Tf(x_1) - Tf(x_2)| = \left| \int_0^1 K(x_1, y)f(y) \, dy - \int_0^1 K(x_2, y)f(y) \, dy \right|$$

$$= \left| \int_0^1 (K(x_1, y) - K(x_2, y))f(y) \, dy \right|$$

$$\leq \int_0^1 |K(x_1, y) - K(x_2, y)| |f(y)| \, dy$$

$$\leq \int_0^1 \frac{\varepsilon}{\|f\|_u} |f(y)| \, dy$$

$$\leq \int_0^1 \frac{\varepsilon}{\|f\|_u} \|f\|_u \, dy$$

$$= \int_0^1 \varepsilon \, dy$$

$$= \varepsilon$$

for all $x_1, x_2 \in [0, 1]$ with $|x_1 - x_2| < \delta$. This shows that $Tf \in C([0, 1])$. Now let $\varepsilon \in (0, \infty)$ and choose $\delta \in (0, \infty)$ such that $\|K(z_1) - K(z_2)\| < \varepsilon$ for all $z_1, z_2 \in [0, 1]^2$ with $|z_1 - z_2| < \delta$. Then $|K(z_1) - K(z_2)| < \varepsilon/\|f\|_u$ for all $f \in C([0, 1])$ with $0 < \|f\|_u \leq 1$ and $z_1, z_2 \in [0, 1]^2$ with $|z_1 - z_2| < \delta$, so $\{Tf \mid \|f\|_u \leq 1\}$ is equicontinuous by the previous calculation and the fact that $|0 - 0| < \varepsilon$. Moreover, if $x \in [0, 1]$ and $f \in C([0, 1])$ with $\|f\|_u \leq 1$ then

$$|Tf(x)| = \left| \int_0^1 K(x, y)f(y) \, dy \right| \leq \int_0^1 |K(x, y)||f(y)| \, dy \leq \int_0^1 |K(x, y)| \, dy,$$

so $\{Tf \mid \|f\|_u \leq 1\}$ is pointwise bounded, and thus precompact by the Arzelà-Ascoli theorem.

64. Let $\varepsilon \in (0, \infty)$ and define $\delta := \sqrt[\alpha]{\varepsilon}$. If $f \in \mathcal{F} := \{f \in C(X) \mid \|f\|_u \leq 1 \text{ and } N_\alpha(f) \leq 1\}$ then

$$|f(x) - f(y)| \leq N_\alpha(f)\rho(x, y) \leq \rho(x, y) \leq \delta^\alpha = \varepsilon$$
for all \(x, y \in X\) with \(\rho(x, y) < \delta\). This implies that \(\mathcal{F}\) is equicontinuous. Moreover, if \(f \in \mathcal{F}\) and \(x \in X\) then 
\[
|f(x)| \leq \|f\|_u \leq 1,
\]
so \(\mathcal{F}\) is pointwise bounded and hence precompact, by the Arzelà-Ascoli theorem. Now let \(\langle f_n \rangle_{n=1}^\infty\) be a sequence in \(\mathcal{F}\) which converges in \(C(X)\) to some \(f \in C(X)\). If \(x \in X\) then \(\langle f_n(x) \rangle_{n=1}^\infty\) is a sequence in \([-1, 1]\) which converges to \(f(x)\), so \(|f(x)| \leq 1\) and hence \(\|f\|_u \leq 1\). Similarly, if \(x, y \in X\) then \(\langle f_n(x) - f_n(y) \rangle_{n=1}^\infty\) converges to \(f(x) - f(y)\), and hence \(|f(x) - f(y)| \leq \rho(x, y)^n\). This implies that \(f \in \mathcal{F}\), so \(\mathcal{F}\) is closed, thus compact.

68. Define \(\mathcal{E} := \{(x, y) \mapsto f(x)g(y) \mid f \in C(X), g \in C(Y)\}\). The collection \(\mathcal{A}\) of finite sums of elements of \(\mathcal{E}\) is the algebra generated by \(\mathcal{E}\), because this algebra contains \(\mathcal{A}\), and \(\mathcal{A}\) is an algebra. Indeed, if \(f_1, f_2, \ldots, f_n, h_1, h_2, \ldots, h_m \in C(X)\) and \(g_1, g_2, \ldots, g_n, k_1, k_2, \ldots, k_m \in C(Y)\), then
\[
\left( \sum_{i=1}^n f_i g_i \right) \left( \sum_{j=1}^m h_j k_j \right) = \sum_{i=1}^n \sum_{j=1}^m f_i g_i h_j k_j = \sum_{i=1}^n \sum_{j=1}^m (f_i h_j)(g_i k_j) \in \mathcal{A}.
\]

Note that \(\mathcal{A}\) is closed under complex conjugation, because if \(f_1, f_2, \ldots, f_n \in C(X)\) and \(g_1, g_2, \ldots, g_n \in C(Y)\) then
\[
\sum_{i=1}^n f_i g_i = \sum_{i=1}^n \overline{f_i} \overline{g_i} \in \mathcal{A}.
\]

Because complex conjugation is continuous, \(\overline{\mathcal{A}}\) is also closed under complex conjugation. If \((x_1, y_1), (x_2, y_2) \in X \times Y\) and \((x_1, y_1) \neq (x_2, y_2)\), then \(x_1 \neq x_2\) or \(y_1 \neq y_2\). In the former case, there exists \(f \in C(X)\) with \(f(x_1) \neq f(x_2)\) because \(X\) is normal, and hence \(f \cdot 1 \in \mathcal{A}\) separates \((x_1, y_1)\) from \((x_2, y_2)\). The latter case is similar, and we conclude that \(\overline{\mathcal{A}}\) separates points. Since \(X \times Y\) is compact and Hausdorff, and \(\mathcal{A}\) contains the constant functions, the complex Stone-Weierstraß theorem implies that \(\overline{\mathcal{A}} = C(X \times Y)\).

69. Let \(\mathcal{A}\) and \(\mathcal{B}\) be the subalgebras of \(C(X)\) and \(C(X, \mathbb{R})\), respectively, generated by the coordinate maps and the constant function 1. Then \(\mathcal{A} = \text{span}_C(\mathcal{B})\), because \(\mathcal{A}\) contains \(\mathcal{B}\) (note that \(\mathcal{A} \cap C(X, \mathbb{R})\) is a subalgebra of \(C(X, \mathbb{R})\)) and \(\text{span}_C(\mathcal{B})\) is an algebra. Indeed, if \(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in \mathbb{C}\) and \(f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_m \in \mathcal{B}\) then
\[
\left( \sum_{i=1}^n a_i f_i \right) \left( \sum_{j=1}^m b_j g_j \right) = \sum_{i=1}^n \sum_{j=1}^m a_i f_i b_j g_j = \sum_{i=1}^n \sum_{j=1}^m (a_i b_j)(f_i g_j) \in \text{span}_C(\mathcal{B}).
\]

Therefore \(\mathcal{A}\) is closed under complex conjugation, because if \(a_1, a_2, \ldots, a_n \in \mathbb{C}\) and \(f_1, f_2, \ldots, f_n \in \mathcal{B}\) then
\[
\sum_{i=1}^n a_i f_i = \sum_{i=1}^n \overline{a_i} \overline{f_i} = \sum_{i=1}^n \overline{a_i} f_i \in \mathcal{A}.
\]

Because complex conjugation is continuous, \(\overline{\mathcal{A}}\) is also closed under complex conjugation. If \(x_1, x_2 \in X\) and \(x_1 \neq x_2\), then \(\pi_\alpha(x_1) \neq \pi_\alpha(x_2)\) for some \(\alpha \in \mathcal{A}\), and hence \(\overline{\mathcal{A}}\) separates points. By Tychonoff’s theorem \(X\) is compact, and it is also Hausdorff because \([0, 1]\) is Hausdorff. Since \(\mathcal{A}\) contains the constant functions, the complex Stone-Weierstraß theorem implies that \(\overline{\mathcal{A}} = C(X)\).

70. (a) By definition \(h(\mathcal{F}) = \cap_{f \in \mathcal{F}} f^{-1}(\{0\})\), which is closed because \(\{0\}\) is closed in \(\mathbb{R}\).

(b) Clearly \(0 \in k(E)\). If \(f, g \in k(E)\) and \(a \in \mathbb{R}\) then \(af(x) + bg(x) = 0\) for all \(x \in E\), so \(af + bg \in k(E)\). Moreover, if \(f \in k(E)\) and \(g \in C(X, \mathbb{R})\) then \(f(x)g(x) = 0\) for all \(x \in E\), so \(fg \in k(E)\). This shows that \(k(E)\) is an ideal of \(C(X, \mathbb{R})\). For each \(x \in E\) the coordinate map \(\pi_x : C(X, \mathbb{R}) \to \mathbb{R}\) is (Lipschitz) continuous (relative to the uniform norm on \(C(X, \mathbb{R})\)), so \(k(E) = \cap_{x \in E} e_x^{-1}(\{0\})\) is closed.
(c) Clearly $E \subseteq h(k(E))$, and hence $\overline{E} \subseteq h(k(E))$. If $x \in \overline{E}$ then, since $X$ is normal, there exists $f \in C(X, \mathbb{R})$ such that $f(\overline{E}) = 0$ and $f(x) = 1$. It follows that $x \notin h(k(E))$, because $f \in k(E)$ but $f(x) \neq 0$. Thus $h(k(E)) = \overline{E}$.

(d) Clearly $J \subseteq k(h(J))$, and hence $\overline{J} \subseteq k(h(J))$. Define $U := X \setminus h(J)$, and let $x \in U$. Then $\{x\}$ and $U^c$ are closed and disjoint, so there exist disjoint open neighbourhoods $V, W \subseteq X$ of $x$ and $U^c$. It follows that $W^c \subseteq U$ is a compact neighbourhood of $x$, since $X$ is compact and $V \subseteq W^c$. This shows that $U$ is locally compact. Define $J := \{f|_U \mid f \in J\}$. If $f \in J$ and $\varepsilon \in (0, \infty)$ then $\{x \in U \mid |f(x)| \geq \varepsilon\} = \{x \in X \mid f(x) \geq \varepsilon\}$ because $f(x) = 0$ for all $x \in h(J)$. Since this set is closed (thus compact), $f|_U \in C_0(U)$. Therefore $J$ is a closed subalgebra of $C_0(U, \mathbb{R})$ (because function restriction respects pointwise addition and multiplication, and uniform limits). If $x, y \in U$ and $x \neq y$, there exists $f \in J$ such that $f(x) \neq 0$. Since $X$ is normal there exists $g \in C(X, \mathbb{R})$ such that $g(x) = 1$ and $g(y) = 0$. Then $gf \in \mathcal{J}$ and hence $(gf)|_U \in \mathcal{J}$, so $\mathcal{J}$ separates points. Moreover, there is no $x \in U$ such that $f(x) = 0$ for all $f \in \mathcal{J}$. By the Stone-Weierstraß theorem, it follows that $\mathcal{J} = C(U, \mathbb{R})$ or $\mathcal{J} = C_0(U, \mathbb{R})$, depending on whether $U$ is closed or not. If $f \in k(h(J))$ then $f|_U \in C_0(U, \mathbb{R})$ for the same reason that $J \subseteq C_0(U, \mathbb{R})$, so $f|_U = g|_U$ for some $g \in \mathcal{J}$, in which case $f = g$ because $f(x) = g(x) = 0$ for all $x \in h(J)$. This shows that $k(h(J)) \subseteq \mathcal{J}$, and hence $k(h(J)) = \mathcal{J}$.

(e) The previous two exercises show that $k$ is a bijection from the closed subsets of $X$ onto the closed ideals of $C(X, \mathbb{R})$.

76. Let $\mathcal{B}$ be a countable base for the topology on $X$. Since $X$ is normal, for each $U, V \in \mathcal{B}$ such that $V \subseteq U$, there exists $f_{U,V} \in C(X, [0,1])$ which is 0 on $U^c$ and 1 on $V$. Clearly $\mathcal{F} := \{f_{U,V} \mid U, V \in \mathcal{B} \text{ and } V \subseteq U\}$ is countable. Let $C \subseteq X$ be closed and let $x \in C^c$. There exists $U \in \mathcal{B}$ such that $x \in U \subseteq C^c$, and by normality there exist disjoint open sets $U', V' \subseteq X$ such that $U^c \subseteq U'$ and $x \in V'$. Then $x \in V \subseteq V'$ for some $V \in \mathcal{B}$, and $V \subseteq V' \subseteq (U')^c \subseteq U$. Now $C \subseteq U^c$, so $f_{U,V}(C) = \{0\}$, while $f_{U,V}(x) = 1$ because $x \in V$. This shows that $\mathcal{F}$ separates points and closed sets.

77. Clearly $\rho$ maps into $[0,1]$. If $x \in X$, then $\rho(x, x) = 0$ by definition. Given $x, y \in X$ with $x \neq y$, there exists $n \in \mathbb{N}$ such that $x_n \neq y_n$, in which case $\rho(x, y) \geq 2^{-n} \rho_n(x_n, y_n) > 0$. Clearly $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$. Moreover,

$$\rho(x, z) = \sum_{n=1}^{\infty} 2^{-n} \rho_n(x_n, z_n) \leq \sum_{n=1}^{\infty} 2^{-n} (\rho_n(x_n, y_n) + \rho(y_n, z_n)) = \sum_{n=1}^{\infty} 2^{-n} \rho_n(x_n, y_n) + \sum_{n=1}^{\infty} 2^{-n} \rho(y_n, z_n) = \rho(x, y) + \rho(y, z)$$

for all $x, y, z \in X$. This shows that $\rho$ is a metric.

Let $n \in \mathbb{N}$ and $U \subseteq X_n$ be open. Given $x \in \pi_n^{-1}(U)$, there exists $r \in (0, \infty)$ such that $B_r(x_n) \subseteq U$. If $y \in B_{2^{-n}r}(x)$ then $\rho(x, y) < 2^{-n} r$, and in particular $\rho_n(x_n, y_n) < r$, which implies that $y_n \in U$ and hence $y \in \pi_n^{-1}(U)$. This shows that $\pi_n^{-1}(U)$ is open in $(X, \rho)$, and hence every member of the product topology on $X$ is open in $(X, \rho)$.

Conversely, let $U \subseteq X$ be open in $(X, \rho)$. Given $x \in U$, there exists $r \in (0, \infty)$ such that $B_r(x) \subseteq U$. Choose $N \in \mathbb{N}$ so that $2^{1-N} \leq r$. For each $n \in \{1, 2, \ldots, N\}$ define $U_n := B_{r/2}(x_n)$, and set $V := \bigcap_{n=1}^{N} \pi_n^{-1}(U_n)$. If $y \in V$ then

$$\rho(x, y) = \sum_{n=1}^{N} 2^{-n} \rho(x_n, y_n) + \sum_{n=N+1}^{\infty} 2^{-n} \rho(x_n, y_n) < \sum_{n=1}^{N} 2^{-n-1} r + \sum_{n=N+1}^{\infty} 2^{-n} < 2^{-1} r + 2^{-N} \leq r,$$

so $y \in U$. Since $x \in V$, this shows that $U$ is open in the product topology. Therefore $(X, \rho)$ has the product topology.